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A Fully Coupled Micro/Macro Theory for Thermo-Electro-Magneto-Elasto-Plastic Composite Laminates

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Abstract

This paper presents a micro/macro theory for determining the coupled thermo-electro-magneto-elasto-plastic behavior of arbitrary composite laminates. Two models are considered. The first is the electo-magnetic generalized method of cells (EMGMC) (Aboudi, 2000) micromechanics model. EMGMC has been completely reformulated to improve its computational efficiency and has been extended to admit arbitrary anisotropic local material behavior (in terms of the thermal response, mechanical response, electric response, magnetic response, as well as the coupling behavior) and inelastic local material behavior. The second model is classical lamination theory, which has also been extended for arbitrary anisotropic material behavior and electro-magnetic effects. The end result is a coupled theory that employs EMGMC to provide the homogenized behavior of the composite plies that constitute the thermo-electro-magnetic laminate. Sample results that illustrate many of the unique aspects of the theory are presented.

1. Introduction

The phenomenon of coupling between the thermo-mechanical behavior of materials and the electro-magnetic behavior of materials has been reported since the 19th century. By the middle of the 20th century, piezoelectric materials were finding their first applications in hydrophones. In the last two decades, the concept of electro-magnetic composite materials has arisen. Such composites can exhibit field coupling that is not present in any of the monolithic constituent materials. With applications in ultrasonic imaging devices, sensors, actuators, transducers, and many other emerging components, there is a strong need for theories that can predict the coupled response of these so call "smart" materials and composites, as well as structures composed of them.

The basic concepts of piezoelectricity, as well as detailed discussions of applications of piezoelectric materials, are available in Gandhi and Thompson (1992) and Uchino (1997). Magnetoelasticity is outlined in the works of Parton and Krudryavtsev (1988) and Krudryavtsev et al. (1990). The application of piezoelectric materials in plate structures was examined by Tiersten (1969) and Tauchert (1992). The former investigation included, within classical lamination theory, the ability to model piezoelectric plies with a specific polarization orientation. For general structures, the commercial finite element code ABAQUS allows piezoelectric analysis through a number of continuum and truss elements (ABAQUS, 2000).

Micromechanics theories, which allow the behavior of a composite to be determined from the behavior of the constituents and their arrangement, have also been extended to included electro-magnetic effects. Early work in this arena was done by Newnham et al. (1978) using a mechanics of materials approach. A piezoelectric concentric cylinder model was presented by Grekov et al. (1989), while Carman et al. (1995) included piezoelectric and piezomagnetic effects within a concentric cylinder approach. A good deal of relevant work in the area of micromechanics has also been done by Dunn and co-workers. Dunn and Taya (1993) and Dunn (1993) included piezoelectric effects and pyroelectric effects within several micromechanics theories, including the dilute approximation, the differential scheme, the self-consistent method, and the Mori-Tanaka (1973) mean field method. Li and Dunn (1998) extended this work by incorporating piezomagnetic effects within the Mori-Tanaka model. The electro-magneto-elastic Mori-Tanaka method has also been the subject of investigations by Wu and Huang (2000) and Huang et al. (2000).

Recently, the method of cells approach, presented in its original form by Aboudi (1989), in its generalized form by Paley and Aboudi (1992), and its triply periodic generalized form by Aboudi (1995), has been extended to include thermo-electric effects (Aboudi, 1998) and thermo-electro-magnetic effects (Aboudi, 2000). These efforts focused on determining the effective thermo-electro-magneto-elastic properties of composites using the triply periodic version of the generalized method of cells (GMC), and illustrated excellent agreement between this electro-magnetic GMC (EMGMC) and the Mori-Tanaka method results presented by Dunn and co-workers.

The objective of this paper is to present the equations and framework for a general coupled thermo-electro-magneto-elasto-plastic theory at both the composite and laminate levels. GMC has been chosen as the micro scale model in order to represent the behavior of the composite from the level of the individual constituents to the level of the homogenized effective material. To model the behavior of the composite laminate, classical lamination theory (see Jones (1975) and Herakovich (1998)) has been employed. The presented framework enables the use of GMC as an embedded constitutive model to represent the local composite material response at through-thickness integration points within the plies of the laminate.

The present micro/macro theory is kept as general as possible in order to maximize its applicability. In pursuing this generality, several significant alterations of and extensions to GMC and lamination theory were required. First, due to the fact that GMC serves as a micro scale model in the present framework and can be used many times to represent points within a laminate, the computational efficiency of the model is important. Thus, EMGMC has been reformulated (as has been done for GMC by Pindera and Bednarcyk (1999) and Bednarcyk and Pindera (2000)) to significantly decrease its number of unknown quantities (i.e., degrees of freedom). Further, inelastic strains have been included in the reformulation that enable inclusion of arbitrary viscoplastic constitutive models on the level of the constituents. Finally, fully anisotropic local material behavior (in terms of the mechanical, piezoelectric, piezomagnetic, and all coupling coefficients) has been preserved. This is important for use within lamination theory so that the layers may be composed of plies with arbitrarily oriented poling directions.

Second, the present thermo-electro-magneto-elasto-plastic lamination theory has been extended (with respect to the presentation of Tauchert (1992)) to include magnetic and inelastic terms. As in the GMC reformulation, the anisotropic material behavior is preserved. Tauchert (1992) assumed orthorhombic crystal symmetry for the laminate plies, with a fixed through-thickness poling direction. This is a severe limitation in that, for example, a ply containing piezoelectric wires with an in-plane orientation would not be admitted by the theory.

A final unique aspect of the present framework arises from its multi-scale nature; the ability to analyze laminates that exhibit inelastic behavior. Most inelastic constitutive models are formulated to represent isotropic materials (e.g., incremental plasticity (Mendelson (1968)), Bodner-Partom viscoplasticity (Chan et al. (1988))). Thus, a structural analysis that includes such inelastic constitutive models must localize to the level of the isotropic inelastic constituents. The present framework allows this localization to occur, and through GMC, the local inelastic strains are homogenized to give effective inelastic strains for the composite material. The equations of lamination theory then homogenize the composite inelastic strains to yield the required laminate scale inelastic force and moment resultants. Inclusion of inelasticity within the present micro/macro theory enables analysis of thermo-electro-magnetic laminates containing metals. This capability is illustrated in the Results section of this paper as results are given for a hybrid laminate composed of both smart composite plies and metal matrix composite plies.

2. The Anisotropic Electro-Magnetic Generalized Method of Cells (EMGMC) Reformulation

As mentioned in the Introduction, this work follows that of Aboudi (2000), who presented the original formulation of the three-dimensional electro-magnetic GMC (EMGMC). Further, the reformulation follows that performed by Bednarcyk and Pindera (2000) for the three-dimensional thermo-elasto-plastic GMC. The geometry of the triply periodic GMC repeating unit cell is shown in Fig. 1, where the parallelepiped repeating unit cell is composed of an arbitrary number of parallelepiped subcells, each of which may be a distinct anisotropic material.

Allowing for complete anisotropy of each subcell material, the subcell constitutive equation is given by,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ D_1 \\ D_2 \\ D_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}^{(\alpha\beta\gamma)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & e_{11} & e_{21} & e_{31} & q_{11} & q_{21} & q_{31} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & e_{12} & e_{22} & e_{32} & q_{12} & q_{22} & q_{32} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} & e_{13} & e_{23} & e_{33} & q_{13} & q_{23} & q_{33} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} & e_{14} & e_{24} & e_{34} & q_{14} & q_{24} & q_{34} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} & e_{15} & e_{25} & e_{35} & q_{15} & q_{25} & q_{35} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} & e_{16} & e_{26} & e_{36} & q_{16} & q_{26} & q_{36} \\ e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & -\kappa_{11} & -\kappa_{12} & -\kappa_{13} & -a_{11} & -a_{12} & -a_{13} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} & -\kappa_{12} & -\kappa_{22} & -\kappa_{23} & -a_{21} & -a_{22} & -a_{23} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & -\kappa_{13} & -\kappa_{23} & -\kappa_{33} & -a_{31} & -a_{32} & -a_{33} \\ q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} & -a_{11} & -a_{21} & -a_{31} & -\mu_{11} & -\mu_{12} & -\mu_{13} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} & q_{26} & -a_{12} & -a_{22} & -a_{32} & -\mu_{12} & -\mu_{22} & -\mu_{23} \\ q_{31} & q_{32} & q_{33} & q_{34} & q_{35} & q_{36} & -a_{13} & -a_{23} & -a_{33} & -\mu_{13} & -\mu_{23} & -\mu_{33} \end{bmatrix}^{(\alpha\beta\gamma)} \begin{bmatrix} \varepsilon_{11} - \varepsilon_{11}^I - \varepsilon_{11}^T \\ \varepsilon_{22} - \varepsilon_{22}^I - \varepsilon_{22}^T \\ \varepsilon_{33} - \varepsilon_{33}^I - \varepsilon_{33}^T \\ 2\varepsilon_{23} - 2\varepsilon_{23}^I - 2\varepsilon_{23}^T \\ 2\varepsilon_{13} - 2\varepsilon_{13}^I - 2\varepsilon_{13}^T \\ 2\varepsilon_{12} - 2\varepsilon_{12}^I - 2\varepsilon_{12}^T \\ -E_1 - E_1^T \\ -E_2 - E_2^T \\ -E_3 - E_3^T \\ -H_1 - H_1^T \\ -H_2 - H_2^T \\ -H_3 - H_3^T \end{bmatrix}^{(\alpha\beta\gamma)} \quad (1)$$

where $\sigma_{ij}^{(\alpha\beta\gamma)}$ are the stress components, $D_k^{(\alpha\beta\gamma)}$ are the electric displacement components, $B_k^{(\alpha\beta\gamma)}$ are the magnetic flux density components, $\varepsilon_{ij}^{(\alpha\beta\gamma)}$ are the total strain components, $\varepsilon_{ij}^{I(\alpha\beta\gamma)}$ are the inelastic strain components, $\varepsilon_{ij}^{T(\alpha\beta\gamma)}$ are the thermal strain components, $E_k^{(\alpha\beta\gamma)}$ are the electric field components, $E_k^{T(\alpha\beta\gamma)}$ are the thermo-electric field components, $H_k^{(\alpha\beta\gamma)}$ are the magnetic field components, $H_k^{T(\alpha\beta\gamma)}$ are the thermo-magnetic field components, $C_{ij}^{(\alpha\beta\gamma)}$ are the material stiffness components, $e_{kj}^{(\alpha\beta\gamma)}$ are the material piezoelectric components, $q_{kj}^{(\alpha\beta\gamma)}$ are the material piezomagnetic components, $\kappa_{ij}^{(\alpha\beta\gamma)}$ are the material dielectric components, $a_{ij}^{(\alpha\beta\gamma)}$ are the material magnetoelectric components, and $\mu_{ij}^{(\alpha\beta\gamma)}$ are the material magnetic permeability components of a given subcell (denoted by the indices $\alpha\beta\gamma$). The thermal strain and thermal field components are related to a change in temperature from a given reference temperature (i.e., ΔT) by,

$$\begin{aligned} &\left[\varepsilon_{11}^T \quad \varepsilon_{22}^T \quad \varepsilon_{33}^T \quad 2\varepsilon_{23}^T \quad 2\varepsilon_{13}^T \quad 2\varepsilon_{12}^T \quad E_1^T \quad E_2^T \quad E_3^T \quad H_1^T \quad H_2^T \quad H_3^T\right]^{(\alpha\beta\gamma)} = \\ &\left[\alpha_{11} \quad \alpha_{22} \quad \alpha_{33} \quad \alpha_{23} \quad \alpha_{13} \quad \alpha_{12} \quad \zeta_1 \quad \zeta_2 \quad \zeta_3 \quad \psi_1 \quad \psi_2 \quad \psi_3\right]^{(\alpha\beta\gamma)} \Delta T \end{aligned} \quad (2)$$

where $\alpha_{ij}^{(\alpha\beta\gamma)}$ are the subcell material coefficients of thermal expansion (CTEs), $\zeta_k^{(\alpha\beta\gamma)}$ are the subcell material pyroelectric constants, and $\psi_k^{(\alpha\beta\gamma)}$ are the subcell material pyromagnetic constants.

Equation (1) can be rewritten as,

$$\begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{D} \\ \mathbf{B} \end{bmatrix}^{(\alpha\beta\gamma)} = [\mathbf{Z}]^{(\alpha\beta\gamma)} \begin{bmatrix} \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^I - \boldsymbol{\varepsilon}^T \\ -\mathbf{E} - \mathbf{E}^T \\ -\mathbf{H} - \mathbf{H}^T \end{bmatrix}^{(\alpha\beta\gamma)} \tag{3}$$

where $\boldsymbol{\sigma}$, $\mathbf{D}$, $\mathbf{B}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^I$, $\boldsymbol{\varepsilon}^T$, $\mathbf{E}$, $\mathbf{E}^T$, $\mathbf{H}$, and $\mathbf{H}^T$ are vectors containing the subcell stress, electric displacement, magnetic flux density, total strain, inelastic strain, thermal strain, electric field, thermo-electric field, magnetic field, and thermo-magnetic field components, respectively. $\mathbf{Z}$ is the subcell 12×12 electro-magneto-elastic coefficient matrix. Rearranging the terms in eq. (3) yields,

$$\begin{bmatrix} \boldsymbol{\varepsilon} \\ -\mathbf{E} \\ -\mathbf{H} \end{bmatrix}^{(\alpha\beta\gamma)} = [\mathbf{ZI}]^{(\alpha\beta\gamma)} \begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{D} \\ \mathbf{B} \end{bmatrix}^{(\alpha\beta\gamma)} + \begin{bmatrix} \boldsymbol{\varepsilon}^I \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}^{(\alpha\beta\gamma)} + \begin{bmatrix} \boldsymbol{\varepsilon}^T \\ \mathbf{E}^T \\ \mathbf{H}^T \end{bmatrix}^{(\alpha\beta\gamma)} \tag{4}$$

where $\mathbf{ZI}$ is the inverse of the matrix $\mathbf{Z}$ and $\mathbf{0}$ represents vector of length three containing all zeros.

As discussed by Pindera and Bednarcyk (1999) and Bednarcyk and Pindera (2000), due to the inherent lack of coupling between normal and shear stress and strain components within the GMC framework, not all subcell stress components are unique. The application of interfacial traction continuity (in an average sense) between subcells results in the constancy of normal stress components in particular columns of subcells throughout the repeating unit cell and constancy of shear stress components in particular layers of subcells (see Bednarcyk and Pindera, 2000). Similarly, by imposing continuity of the normal interfacial electric displacement components and the normal interfacial magnetic flux density components results in the constancy of certain electric displacement components and certain magnetic flux density components in particular rows of subcells. By taking advantage of this feature of GMC during the formulation, the computational efficiency of the model is dramatically improved. The unique subcell stress, electric displacement, and magnetic flux density components are,

$$\sigma_{11}^{(\beta\gamma)},\ \sigma_{22}^{(\alpha\gamma)},\ \sigma_{33}^{(\alpha\beta)},\ \sigma_{23}^{(\alpha)},\ \sigma_{13}^{(\beta)},\ \sigma_{12}^{(\gamma)},\ D_1^{(\beta\gamma)},\ D_2^{(\alpha\gamma)},\ D_3^{(\alpha\beta)},\ B_1^{(\beta\gamma)},\ B_2^{(\alpha\gamma)},\ B_3^{(\alpha\beta)} \tag{5}$$

Equation (5) embodies the traction, normal electric displacement, and normal magnetic flux density continuity conditions of EMGMC. For instance, one traction continuity condition requires that $\sigma_{11}^{(\hat{\alpha}\beta\gamma)} = \sigma_{11}^{(\alpha\beta\gamma)}$ for all α and $\hat{\alpha}$. By stating that only the components $\sigma_{11}^{(\beta\gamma)}$ are unique, this condition is satisfied.

Aboudi (1998, 2000) derived the following twelve equations establishing the continuity of displacements, electric potential, and magnetic potential (in an average sense) between subcell interfaces,

$$
\begin{aligned}
&\sum_{\alpha} d_{\alpha}\varepsilon_{11}^{(\alpha\beta\gamma)} = d\bar{\varepsilon}_{11} && \sum_{\alpha} d_{\alpha}E_{1}^{(\alpha\beta\gamma)} = d\bar{E}_{1} \\
&\sum_{\beta} h_{\beta}\varepsilon_{22}^{(\alpha\beta\gamma)} = h\bar{\varepsilon}_{22} && \sum_{\beta} h_{\beta}E_{2}^{(\alpha\beta\gamma)} = h\bar{E}_{2} \\
&\sum_{\gamma} l_{\gamma}\varepsilon_{33}^{(\alpha\beta\gamma)} = l\bar{\varepsilon}_{33} && \sum_{\gamma} l_{\gamma}E_{3}^{(\alpha\beta\gamma)} = l\bar{E}_{3} \\
&\sum_{\beta}\sum_{\gamma} h_{\beta}l_{\gamma}\varepsilon_{23}^{(\alpha\beta\gamma)} = hl\bar{\varepsilon}_{23} && \sum_{\alpha} d_{\alpha}H_{1}^{(\alpha\beta\gamma)} = d\bar{H}_{1} \\
&\sum_{\alpha}\sum_{\gamma} d_{\alpha}l_{\gamma}\varepsilon_{13}^{(\alpha\beta\gamma)} = dl\bar{\varepsilon}_{13} && \sum_{\beta} h_{\beta}H_{2}^{(\alpha\beta\gamma)} = h\bar{H}_{2} \\
&\sum_{\alpha}\sum_{\beta} d_{\alpha}h_{\beta}\varepsilon_{12}^{(\alpha\beta\gamma)} = dh\bar{\varepsilon}_{12} && \sum_{\gamma} l_{\gamma}H_{3}^{(\alpha\beta\gamma)} = l\bar{H}_{3}
\end{aligned}
\tag{6}
$$

where the overbar terms are global or average components, which represent the behavior of the homogenized repeating unit cell, and a summation over an index α, β, or γ implies summation over all values of that index, e.g., $\sum_{\alpha} = \sum_{\alpha=1}^{N_\alpha}$, with N_α, N_β, and N_γ being the number of subcells within the repeating unit cell in the x_1, x_2, and x_3 coordinate directions, respectively (see Fig. 1).

The next step is to substitute for the subcell strain, electric field, and magnetic field components in eq. (6) using the rearranged subcell constitutive equation (4) while retaining only the unique subcell stress, electric displacement, and magnetic flux density components given in eq. (5). This procedure results in 12 equations of the form,

$$
\begin{aligned}
&\sum_{\alpha} d_{\alpha}\, ZI_{1,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)} \\
&+\sum_{\alpha} d_{\alpha}\, ZI_{1,7}^{(\alpha\beta\gamma)}D_{1}^{(\beta\gamma)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,8}^{(\alpha\beta\gamma)}D_{2}^{(\alpha\gamma)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,9}^{(\alpha\beta\gamma)}D_{3}^{(\alpha\beta)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,10}^{(\alpha\beta\gamma)}B_{1}^{(\beta\gamma)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,11}^{(\alpha\beta\gamma)}B_{2}^{(\alpha\gamma)} + \sum_{\alpha} d_{\alpha}\, ZI_{1,12}^{(\alpha\beta\gamma)}B_{3}^{(\alpha\beta)} \\
&= d\bar{\varepsilon}_{11} - \sum_{\alpha} d_{\alpha}\varepsilon_{11}^{I(\alpha\beta\gamma)} - \sum_{\alpha} d_{\alpha}\varepsilon_{11}^{T(\alpha\beta\gamma)} \qquad \text{for all } \beta,\gamma
\end{aligned}
\tag{7}
$$

where $ZI_{i,j}^{(\alpha\beta\gamma)}$ denotes the components of the matrix $\mathbf{ZI}$ of subcell $(\alpha\beta\gamma)$. The remaining eleven equations of this form are given in the Appendix. This vital group of 12 equations can be assembled into the following system,

$$
\left[\tilde{\mathbf{G}}\right]\begin{bmatrix} \sigma_{11}^{(\beta\gamma)} \\ \sigma_{22}^{(\alpha\gamma)} \\ \sigma_{33}^{(\alpha\beta)} \\ \sigma_{23}^{(\alpha)} \\ \sigma_{13}^{(\beta)} \\ \sigma_{12}^{(\gamma)} \\ D_1^{(\beta\gamma)} \\ D_2^{(\alpha\gamma)} \\ D_3^{(\alpha\beta)} \\ B_1^{(\beta\gamma)} \\ B_2^{(\alpha\gamma)} \\ B_3^{(\alpha\beta)} \end{bmatrix} = \begin{bmatrix} d\bar{\varepsilon}_{11}^{(\beta\gamma)} - \sum_{\alpha} d_\alpha \varepsilon_{11}^{I(\alpha\beta\gamma)} - \sum_{\alpha} d_\alpha \varepsilon_{11}^{T(\alpha\beta\gamma)} \\ h\bar{\varepsilon}_{22}^{(\alpha\gamma)} - \sum_{\beta} h_\beta \varepsilon_{22}^{I(\alpha\beta\gamma)} - \sum_{\beta} h_\beta \varepsilon_{22}^{T(\alpha\beta\gamma)} \\ l\bar{\varepsilon}_{33}^{(\alpha\beta)} - \sum_{\gamma} l_\gamma \varepsilon_{33}^{I(\alpha\beta\gamma)} - \sum_{\gamma} l_\gamma \varepsilon_{33}^{T(\alpha\beta\gamma)} \\ 2hl\bar{\varepsilon}_{23}^{(\alpha)} - 2\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma \varepsilon_{23}^{I(\alpha\beta\gamma)} - 2\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma \varepsilon_{23}^{T(\alpha\beta\gamma)} \\ 2dl\bar{\varepsilon}_{13}^{(\beta)} - 2\sum_{\alpha}\sum_{\gamma} d_\alpha l_\gamma \varepsilon_{13}^{I(\alpha\beta\gamma)} - 2\sum_{\alpha}\sum_{\gamma} d_\alpha l_\gamma \varepsilon_{13}^{T(\alpha\beta\gamma)} \\ 2dh\bar{\varepsilon}_{12}^{(\gamma)} - 2\sum_{\alpha}\sum_{\beta} d_\alpha h_\beta \varepsilon_{12}^{I(\alpha\beta\gamma)} - 2\sum_{\alpha}\sum_{\beta} d_\alpha h_\beta \varepsilon_{12}^{T(\alpha\beta\gamma)} \\ -d\bar{E}_1^{(\beta\gamma)} - \sum_{\alpha} d_\alpha E_1^{T(\alpha\beta\gamma)} \\ -h\bar{E}_2^{(\alpha\gamma)} - \sum_{\beta} h_\beta E_2^{T(\alpha\beta\gamma)} \\ -l\bar{E}_3^{(\alpha\beta)} - \sum_{\gamma} l_\gamma E_3^{T(\alpha\beta\gamma)} \\ -d\bar{H}_1^{(\beta\gamma)} - \sum_{\alpha} d_\alpha H_1^{T(\alpha\beta\gamma)} \\ -h\bar{H}_2^{(\alpha\gamma)} - \sum_{\beta} h_\beta H_2^{T(\alpha\beta\gamma)} \\ -l\bar{H}_3^{(\alpha\beta)} - \sum_{\gamma} l_\gamma H_3^{T(\alpha\beta\gamma)} \end{bmatrix} \tag{8}
$$

or,

$$
\tilde{\mathbf{G}}\mathbf{T} = \mathbf{f}^M - \mathbf{f}^I - \mathbf{f}^T \tag{9}
$$

Equation (9) represents a system of $N_\alpha + N_\beta + N_\gamma + 3\left(N_\beta N_\gamma + N_\alpha N_\gamma + N_\alpha N_\beta\right)$ linear equations whose solution provides the unique subcell stress, electric displacement, and magnetic flux density components in the subcells of the repeating unit cell given the global strain components, global electric field components, global magnetic field components, subcell inelastic strain components, subcell thermal strain components, subcell thermo-electric field components, and subcell thermo-magnetic field components. The index superscripts in eq. (8) indicate that the vector contains the superscripted components for all values of the superscript indices. For example, the vector $\mathbf{T}$ contains the components, $\sigma_{11}^{(11)}$, $\sigma_{11}^{(12)}$, …, $\sigma_{11}^{(1N_\gamma)}$, $\sigma_{11}^{(21)}$, $\sigma_{11}^{(22)}$ …, $\sigma_{11}^{(2N_\gamma)}$, …, $\sigma_{11}^{(N_\beta N_\gamma)}$, $\sigma_{22}^{(11)}$, …, $\sigma_{22}^{(N_\alpha N_\gamma)}$, …, $B_3^{(N_\alpha N_\beta)}$. As before, the overbars in eq. (8) represent global quantities that apply to the homogenized unit cell, and, as such, do not vary among the subcells. Hence, the superscripts associated with these terms simply indicate that, as in the vector $\mathbf{T}$, the vector $\mathbf{f}^M$ contains these terms repeated for all values of the superscript indices. The general form of the square $\tilde{\mathbf{G}}$ matrix is given in Fig. 2. In general $\tilde{\mathbf{G}}$ may be fully populated, but in many cases it is quite sparse. The sparseness of this matrix is related to the population of the electro-magneto-elastic coefficient matrix, $\mathbf{Z}^{(\alpha\beta\gamma)}$, of the subcells comprising the repeating unit cell, as well as the number of subcells in the repeating unit cell.

Equation (9) is solved for the vector $\mathbf{T}$ to yield mixed thermo-electro-magneto-elasto-plastic concentration equations for the repeating unit cell,

$$
\mathbf{T} = \mathbf{G}\left[\mathbf{f}^M - \mathbf{f}^I - \mathbf{f}^T\right] \tag{10}
$$

These are mixed concentration equations because they provide the local (subcell) stresses, electric displacements, and magnetic flux densities (in the vector $\mathbf{T}$) in terms of the global strain, electric field, and magnetic field components (in the vector $\mathbf{f}^M$) and local inelastic strains, thermal strains, thermo-electric field components, and thermo-magnetic field components (in the vectors $\mathbf{f}^I$ and $\mathbf{f}^T$).

In solving eq. (9), the square matrix $\tilde{\mathbf{G}}$, which is of order $N_\alpha + N_\beta + N_\gamma + 3\left(N_\beta N_\gamma + N_\alpha N_\gamma + N_\alpha N_\beta\right)$, was inverted. As the number of subcells in the repeating unit cell becomes large, this inversion can become computationally intensive. However, in Aboudi's (2000) original formulation of EMGMC, the subcell strain, electric field, and magnetic field components were employed as the basic unknown quantities (as opposed to the stress, electric displacement, and magnetic flux density components employed in the present reformulation). Since all subcell strain, electric field, and magnetic field components are unique, Aboudi's (2000) original formulation resulted in 12 unknown quantities per subcell, for a total of $12 N_\alpha N_\beta N_\gamma$ unknowns. Fig. 3 shows a plot of the number of subcells vs. number of unknowns (i.e., degrees of freedom) for the original formulation of EMGMC and the present reformulation for the case where $N_\alpha = N_\beta = N_\gamma$. For a reasonably-sized 12×12×12 repeating unit cell, the number of unknowns is reduced from 20,736 to 1,332 by employing the present reformulation. A striking improvement in the efficiency of the model, when implemented in a computer code, will also result from this reduction in unknowns.

Equation (10) provides the subcell stress, electric displacement, and magnetic flux density components. Using the subcell constitutive equations, eq. (1), the subcell strain, electric field, and magnetic field components can then be determined. However, the preceding statements presuppose knowledge of the global strain, electric field, and magnetic field components (for the homogenized repeating unit cell) that appear in the vector $\mathbf{f}^M$ of eq. (10). If all of these components are not known, and rather a mixed set of global stress, electric displacement, magnetic flux density, strain, electric field, and magnetic field components is known, the global or effective constitutive equation for the homogenized repeating unit cell is required in order to determine the global components in $\mathbf{f}^M$. The global electro-magneto-elastic coefficient matrix, $\mathbf{Z}^*$, which appears in the global constitutive equation, also provides the effective electro-magneto-elastic properties of the homogenized repeating unit cell. The global (or effective) constitutive equation that is needed is identical in form to the subcell constitutive equations given in eqs. (1) and (3). This equation can be written as,

$$\begin{bmatrix} \bar{\boldsymbol{\sigma}} \\ \bar{\mathbf{D}} \\ \bar{\mathbf{B}} \end{bmatrix} = \left[\mathbf{Z}^*\right] \begin{bmatrix} \bar{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\varepsilon}}^I - \bar{\boldsymbol{\varepsilon}}^T \\ -\bar{\mathbf{E}} - \bar{\mathbf{E}}^T \\ -\bar{\mathbf{H}} - \bar{\mathbf{H}}^T \end{bmatrix} \tag{11}$$

Clearly, complete knowledge of the global constitutive equation involves determination of the global electro-magneto-elastic coefficient matrix, $\mathbf{Z}^*$, the global inelastic strain components, $\bar{\boldsymbol{\varepsilon}}^I$, the global thermal strain components, $\bar{\boldsymbol{\varepsilon}}^T$, the global thermo-electric field components, $\bar{\mathbf{E}}^T$, and the global thermo-magnetic field components, $\bar{\mathbf{H}}^T$.

In order to determine the required components of the global constitutive equation, relations between the global stress, electric displacement, and magnetic flux density components and the corresponding subcell components are employed. In the context of homogenization theory, by definition the global stress, electric displacement, and magnetic flux density components of the homogenized material must equal the volume-weighted sum of their subcell counterparts,

$$
\begin{aligned}
\bar{\sigma}_{ij} &= \frac{1}{dhl}\sum_{\alpha}\sum_{\beta}\sum_{\gamma} d_\alpha h_\beta l_\gamma \sigma_{ij}^{(\alpha\beta\gamma)} \\
\bar{D}_{k} &= \frac{1}{dhl}\sum_{\alpha}\sum_{\beta}\sum_{\gamma} d_\alpha h_\beta l_\gamma D_{k}^{(\alpha\beta\gamma)} \\
\bar{B}_{k} &= \frac{1}{dhl}\sum_{\alpha}\sum_{\beta}\sum_{\gamma} d_\alpha h_\beta l_\gamma B_{k}^{(\alpha\beta\gamma)}
\end{aligned} \tag{12}
$$

Retaining only the unique subcell components, as indicated by eq. (5), eq. (12) simplifies to,

$$
\begin{array}{ccc}
\bar{\sigma}_{11} = \frac{1}{hl}\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma \sigma_{11}^{(\beta\gamma)} & \bar{\sigma}_{22} = \frac{1}{dl}\sum_{\alpha}\sum_{\gamma} d_\alpha l_\gamma \sigma_{22}^{(\alpha\gamma)} & \bar{\sigma}_{33} = \frac{1}{dh}\sum_{\alpha}\sum_{\beta} d_\alpha h_\beta \sigma_{33}^{(\alpha\beta)} \\
\bar{\sigma}_{23} = \frac{1}{d}\sum_{\alpha} d_\alpha \sigma_{23}^{(\alpha)} & \bar{\sigma}_{13} = \frac{1}{h}\sum_{\beta} h_\beta \sigma_{13}^{(\beta)} & \bar{\sigma}_{12} = \frac{1}{l}\sum_{\gamma} l_\gamma \sigma_{12}^{(\gamma)} \\
\bar{D}_{1} = \frac{1}{hl}\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma D_{1}^{(\beta\gamma)} & \bar{D}_{2} = \frac{1}{dl}\sum_{\alpha}\sum_{\gamma} d_\alpha l_\gamma D_{2}^{(\alpha\gamma)} & \bar{D}_{3} = \frac{1}{dh}\sum_{\alpha}\sum_{\beta} d_\alpha h_\beta D_{3}^{(\alpha\beta)} \\
\bar{B}_{1} = \frac{1}{hl}\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma B_{1}^{(\beta\gamma)} & \bar{B}_{2} = \frac{1}{dl}\sum_{\alpha}\sum_{\gamma} d_\alpha l_\gamma B_{2}^{(\alpha\gamma)} & \bar{B}_{3} = \frac{1}{dh}\sum_{\alpha}\sum_{\beta} d_\alpha h_\beta B_{3}^{(\alpha\beta)}
\end{array} \tag{13}
$$

The solutions for the subcell stress, electric displacement, and magnetic flux density components, eq. (10), are substituted into eq. (13) to yield twelve equations of the form,

$$
\begin{aligned}
\bar{\sigma}_{11} = \frac{1}{hl}\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma \Bigg[& d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{\varepsilon}_{33} \\
& + 2hl\sum_{i=1}^{N_\alpha} G\left(R_{\beta\gamma}, i + N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\beta\gamma}, i + N_2 + N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\beta\gamma}, i + N_2 + N_\alpha + N_\beta\right)\bar{\varepsilon}_{12} \\
& - d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i + N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{E}_3 \\
& - d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_2 + N_\beta N_\gamma\right)\bar{H}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{H}_3 \\
& - \sum_{i=1}^{N_1} G\left(R_{\beta\gamma}, i\right) f^{I}(i) - \sum_{i=1}^{N_4} G\left(R_{\beta\gamma}, i\right) f^{T}(i) \Bigg]
\end{aligned} \tag{14}
$$

where $G(i,j)$ refers to the components of the matrix $\mathbf{G}$, $f^{I}(i)$ refers to the components of the vector $\mathbf{f}^{I}$, and $f^{T}(i)$ refers to the components of the vector $\mathbf{f}^{T}$ (see eq. (10)). The additional terms appearing in eq. (14) are,

$$
\begin{array}{cc}
R_{\beta\gamma} = \beta + N_\beta(\gamma - 1) & N_2 = N_\beta N_\gamma + N_\alpha N_\gamma + N_\alpha N_\beta \\
N_1 = N_2 + N_\alpha + N_\beta + N_\gamma & N_4 = N_1 + 2N_2
\end{array} \tag{15}
$$

where once again N_α, N_β, and N_γ are the number of subcells within the repeating unit cell in the three coordinate directions. The remaining eleven equations of this form are given in the Appendix.

The expanded form of the global constitutive equation (11) is,

$$\begin{bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{33} \\ \bar{\sigma}_{23} \\ \bar{\sigma}_{13} \\ \bar{\sigma}_{12} \\ \bar{D}_1 \\ \bar{D}_2 \\ \bar{D}_3 \\ \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{bmatrix} = \begin{bmatrix} C^*_{11} & C^*_{12} & C^*_{13} & C^*_{14} & C^*_{15} & C^*_{16} & e^*_{11} & e^*_{21} & e^*_{31} & q^*_{11} & q^*_{21} & q^*_{31} \\ C^*_{12} & C^*_{22} & C^*_{23} & C^*_{24} & C^*_{25} & C^*_{26} & e^*_{12} & e^*_{22} & e^*_{32} & q^*_{12} & q^*_{22} & q^*_{32} \\ C^*_{13} & C^*_{23} & C^*_{33} & C^*_{34} & C^*_{35} & C^*_{36} & e^*_{13} & e^*_{23} & e^*_{33} & q^*_{13} & q^*_{23} & q^*_{33} \\ C^*_{14} & C^*_{24} & C^*_{34} & C^*_{44} & C^*_{45} & C^*_{46} & e^*_{14} & e^*_{24} & e^*_{34} & q^*_{14} & q^*_{24} & q^*_{34} \\ C^*_{15} & C^*_{25} & C^*_{35} & C^*_{45} & C^*_{55} & C^*_{56} & e^*_{15} & e^*_{25} & e^*_{35} & q^*_{15} & q^*_{25} & q^*_{35} \\ C^*_{16} & C^*_{26} & C^*_{36} & C^*_{46} & C^*_{56} & C^*_{66} & e^*_{16} & e^*_{26} & e^*_{36} & q^*_{16} & q^*_{26} & q^*_{36} \\ e^*_{11} & e^*_{12} & e^*_{13} & e^*_{14} & e^*_{15} & e^*_{16} & -\kappa^*_{11} & -\kappa^*_{12} & -\kappa^*_{13} & -a^*_{11} & -a^*_{12} & -a^*_{13} \\ e^*_{21} & e^*_{22} & e^*_{23} & e^*_{24} & e^*_{25} & e^*_{26} & -\kappa^*_{12} & -\kappa^*_{22} & -\kappa^*_{23} & -a^*_{21} & -a^*_{22} & -a^*_{23} \\ e^*_{31} & e^*_{32} & e^*_{33} & e^*_{34} & e^*_{35} & e^*_{36} & -\kappa^*_{13} & -\kappa^*_{23} & -\kappa^*_{33} & -a^*_{31} & -a^*_{32} & -a^*_{33} \\ q^*_{11} & q^*_{12} & q^*_{13} & q^*_{14} & q^*_{15} & q^*_{16} & -a^*_{11} & -a^*_{21} & -a^*_{31} & -\mu^*_{11} & -\mu^*_{12} & -\mu^*_{13} \\ q^*_{21} & q^*_{22} & q^*_{23} & q^*_{24} & q^*_{25} & q^*_{26} & -a^*_{12} & -a^*_{22} & -a^*_{32} & -\mu^*_{12} & -\mu^*_{22} & -\mu^*_{23} \\ q^*_{31} & q^*_{32} & q^*_{33} & q^*_{34} & q^*_{35} & q^*_{36} & -a^*_{13} & -a^*_{23} & -a^*_{33} & -\mu^*_{13} & -\mu^*_{23} & -\mu^*_{33} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_{11} - \bar{\varepsilon}^I_{11} - \bar{\varepsilon}^T_{11} \\ \bar{\varepsilon}_{22} - \bar{\varepsilon}^I_{22} - \bar{\varepsilon}^T_{22} \\ \bar{\varepsilon}_{33} - \bar{\varepsilon}^I_{33} - \bar{\varepsilon}^T_{33} \\ 2\bar{\varepsilon}_{23} - 2\bar{\varepsilon}^I_{23} - 2\bar{\varepsilon}^T_{23} \\ 2\bar{\varepsilon}_{13} - 2\bar{\varepsilon}^I_{13} - 2\bar{\varepsilon}^T_{13} \\ 2\bar{\varepsilon}_{12} - 2\bar{\varepsilon}^I_{12} - 2\bar{\varepsilon}^T_{12} \\ -\bar{E}_1 - \bar{E}^T_1 \\ -\bar{E}_2 - \bar{E}^T_2 \\ -\bar{E}_3 - \bar{E}^T_3 \\ -\bar{H}_1 - \bar{H}^T_1 \\ -\bar{H}_2 - \bar{H}^T_2 \\ -\bar{H}_3 - \bar{H}^T_3 \end{bmatrix} \quad (16)$$

In eq. (16), the constituents of the global electro-magneto-elastic coefficient matrix, $\mathbf{Z}^*$, of the homogenized repeating unit cell are indicated. C^*_{ij} are the effective stiffness components, e^*_{kj} are the effective piezoelectric components, q^*_{kj} are the effective piezomagnetic components, κ^*_{ij} are the effective dielectric components, a^*_{ij} are the effective magnetoelectric components, and μ^*_{ij} are the effective magnetic permeability components. Comparing the twelve equations of the form of eq. (14) to the global constitutive equation (16), the components of the global electro-magneto-elastic coefficient matrix, $\mathbf{Z}^*$, the global inelastic strain vector, $\bar{\boldsymbol{\varepsilon}}^I$, the global thermal strain vector, $\bar{\boldsymbol{\varepsilon}}^T$, the global thermo-electric field vector, $\bar{\mathbf{E}}^T$, and the global thermo-magnetic field vector, $\bar{\mathbf{H}}^T$, can be readily identified. The expressions for these terms are given in the Appendix.

With the knowledge of these terms in eq. (16), the reformulation of the thermo-electro-magneto-elasto-plastic GMC is complete. Given any admissible state of global mixed stress/strain, electric displacement/electric field, and magnetic flux density/magnetic field for the homogenized material, the unknown global stress/strain, electric displacement/electric field, and magnetic flux density/magnetic field components are determined from eq. (16). Then, eq. (10) provides the local (subcell) stress, electric displacement, and magnetic flux density components, from which the local (subcell) strain, electric field, and magnetic field components can be determined via eq. (1). This represents the complete local/global solution for the repeating unit cell.

The preceding assumes knowledge of the local and global thermal and inelastic terms as well. Given the current (spatially) constant temperature for the repeating unit cell, the subcell thermal strain, thermo-electric field, and thermo-magnetic field components are determined from eq. (2). Then the global thermal strain, thermo-electric field, and thermo-magnetic field components can be determined from the equations in the appendix. The local (subcell) inelastic strains, on the other hand, must be determined from an appropriate local inelastic constitutive model. Typically, such constitutive models provide local inelastic strain increments or rates based on the local stress or strain state, local stress or strain rates, and some internal state variables. This type of model functions seamlessly within the present reformulation of EMGMC as the required local stress and strain fields (as well as the time rates of change of these fields) are known throughout the repeating unit cell. In the presence of inelasticity, the desired state of global stress/strain, electric displacement/electric field, and magnetic flux density/magnetic field must typically be applied in an incremental fashion, and the local increments of inelastic strain (as well as any state variable increments) provided by the local constitutive model must be integrated to provide the local inelastic strains. Once these local inelastic strains are determined, however, the global inelastic strains can be readily obtained from the equations in the Appendix. The particulars of determining the local inelastic strains are clearly associated with the implementation of EMGMC, in conjunction with a specific inelastic constitutive model, within a computer code, rather than the present derivation of the micromechanics theory.

3. Thermo-Electro-Magneto-Elasto-Plastic Lamination Theory

For general presentations of classical lamination theory, the reader is referred to the excellent treatments by Jones (1968) and Herakovich (1998). The present formulation also extends that of Tauchert (1992), which itself extended classical thermo-elastic lamination theory to include piezoelectric terms for laminates including plies with a specific polarization orientation and class. The present treatment is considerably more extensive than the aforementioned presentations in that,

1. The present formulation includes coupled magnetic effects
2. The present formulation is completely general in terms of the material behavior. The lamina may be mechanically monoclinic and of any class electromagnetically with arbitrary poling direction.
3. The present formulation includes inelastic effects that may be modeled locally with an arbitrary constitutive model.

Further, the main objective of the present formulation is to allow the local thermo-electro-magneto-elasto-plastic behavior of the homogenized material at the integration points of the laminate plies to be modeled using the EMGMC theory presented in the previous section.

The geometry of the laminated plate is shown in Fig. 4. At a through-thickness integration point, the full three-dimensional effective thermo-electro-magneto-elasto-plastic constitutive equation (16) is operative for the homogenized material. If attention is limited to the stresses at the integration points, eq. (16) reduces to,

$$\begin{bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{33} \\ \bar{\sigma}_{23} \\ \bar{\sigma}_{13} \\ \bar{\sigma}_{12} \end{bmatrix} = \begin{bmatrix} C^*_{11} & C^*_{12} & C^*_{13} & C^*_{14} & C^*_{15} & C^*_{16} \\ C^*_{12} & C^*_{22} & C^*_{23} & C^*_{24} & C^*_{25} & C^*_{26} \\ C^*_{13} & C^*_{23} & C^*_{33} & C^*_{34} & C^*_{35} & C^*_{36} \\ C^*_{14} & C^*_{24} & C^*_{34} & C^*_{44} & C^*_{45} & C^*_{46} \\ C^*_{15} & C^*_{25} & C^*_{35} & C^*_{45} & C^*_{55} & C^*_{56} \\ C^*_{16} & C^*_{25} & C^*_{36} & C^*_{46} & C^*_{56} & C^*_{66} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_{11} - \bar{\varepsilon}^I_{11} - \bar{\varepsilon}^T_{11} \\ \bar{\varepsilon}_{22} - \bar{\varepsilon}^I_{22} - \bar{\varepsilon}^T_{22} \\ \bar{\varepsilon}_{33} - \bar{\varepsilon}^I_{33} - \bar{\varepsilon}^T_{33} \\ 2\bar{\varepsilon}_{23} - 2\bar{\varepsilon}^I_{23} - 2\bar{\varepsilon}^T_{23} \\ 2\bar{\varepsilon}_{13} - 2\bar{\varepsilon}^I_{13} - 2\bar{\varepsilon}^T_{13} \\ 2\bar{\varepsilon}_{12} - 2\bar{\varepsilon}^I_{12} - 2\bar{\varepsilon}^T_{12} \end{bmatrix} - \begin{bmatrix} e^*_{11} & e^*_{21} & e^*_{31} \\ e^*_{12} & e^*_{22} & e^*_{32} \\ e^*_{13} & e^*_{23} & e^*_{33} \\ e^*_{14} & e^*_{24} & e^*_{34} \\ e^*_{15} & e^*_{25} & e^*_{35} \\ e^*_{16} & e^*_{26} & e^*_{36} \end{bmatrix} \begin{bmatrix} \bar{E}_1 + \bar{E}^T_1 \\ \bar{E}_2 + \bar{E}^T_2 \\ \bar{E}_3 + \bar{E}^T_3 \end{bmatrix} - \begin{bmatrix} q^*_{11} & q^*_{21} & q^*_{31} \\ q^*_{12} & q^*_{22} & q^*_{32} \\ q^*_{13} & q^*_{23} & q^*_{33} \\ q^*_{14} & q^*_{24} & q^*_{34} \\ q^*_{15} & q^*_{25} & q^*_{35} \\ q^*_{16} & q^*_{26} & q^*_{36} \end{bmatrix} \begin{bmatrix} \bar{H}_1 + \bar{H}^T_1 \\ \bar{H}_2 + \bar{H}^T_2 \\ \bar{H}_3 + \bar{H}^T_3 \end{bmatrix} \quad (17)$$

Equation (17) is transformed to the laminate coordinate system, where, despite the engineering matrix notation employed herein, $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\varepsilon}}$, $\bar{\boldsymbol{\varepsilon}}^I$, and $\bar{\boldsymbol{\varepsilon}}^T$ are second-order tensors, $\mathbf{C}^*$ is a fourth-order tensor, $\bar{\mathbf{E}}$, $\bar{\mathbf{E}}^T$, $\bar{\mathbf{H}}$, and $\bar{\mathbf{H}}^T$ are first-order tensors, and $\mathbf{e}^*$ and $\mathbf{q}^*$ are third order tensors, and must be transformed as such. In laminate (x, y, z) coordinates, eq. (17) becomes,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{C}^*_{11} & \bar{C}^*_{12} & \bar{C}^*_{13} & \bar{C}^*_{14} & \bar{C}^*_{15} & \bar{C}^*_{16} \\ \bar{C}^*_{12} & \bar{C}^*_{22} & \bar{C}^*_{23} & \bar{C}^*_{24} & \bar{C}^*_{25} & \bar{C}^*_{26} \\ \bar{C}^*_{13} & \bar{C}^*_{23} & \bar{C}^*_{33} & \bar{C}^*_{34} & \bar{C}^*_{35} & \bar{C}^*_{36} \\ \bar{C}^*_{14} & \bar{C}^*_{24} & \bar{C}^*_{34} & \bar{C}^*_{44} & \bar{C}^*_{45} & \bar{C}^*_{46} \\ \bar{C}^*_{15} & \bar{C}^*_{25} & \bar{C}^*_{35} & \bar{C}^*_{45} & \bar{C}^*_{55} & \bar{C}^*_{56} \\ \bar{C}^*_{16} & \bar{C}^*_{25} & \bar{C}^*_{36} & \bar{C}^*_{46} & \bar{C}^*_{56} & \bar{C}^*_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} - \varepsilon^I_{xx} - \varepsilon^T_{xx} \\ \varepsilon_{yy} - \varepsilon^I_{yy} - \varepsilon^T_{yy} \\ \varepsilon_{zz} - \varepsilon^I_{zz} - \varepsilon^T_{zz} \\ 2\varepsilon_{yz} - 2\varepsilon^I_{yz} - 2\varepsilon^T_{yz} \\ 2\varepsilon_{xz} - 2\varepsilon^I_{xz} - 2\varepsilon^T_{xz} \\ 2\varepsilon_{xy} - 2\varepsilon^I_{xy} - 2\varepsilon^T_{xy} \end{bmatrix} - \begin{bmatrix} \bar{e}^*_{11} & \bar{e}^*_{21} & \bar{e}^*_{31} \\ \bar{e}^*_{12} & \bar{e}^*_{22} & \bar{e}^*_{32} \\ \bar{e}^*_{13} & \bar{e}^*_{23} & \bar{e}^*_{33} \\ \bar{e}^*_{14} & \bar{e}^*_{24} & \bar{e}^*_{34} \\ \bar{e}^*_{15} & \bar{e}^*_{25} & \bar{e}^*_{35} \\ \bar{e}^*_{16} & \bar{e}^*_{26} & \bar{e}^*_{36} \end{bmatrix} \begin{bmatrix} E_x + E^T_x \\ E_y + E^T_y \\ E_z + E^T_z \end{bmatrix} - \begin{bmatrix} \bar{q}^*_{11} & \bar{q}^*_{21} & \bar{q}^*_{31} \\ \bar{q}^*_{12} & \bar{q}^*_{22} & \bar{q}^*_{32} \\ \bar{q}^*_{13} & \bar{q}^*_{23} & \bar{q}^*_{33} \\ \bar{q}^*_{14} & \bar{q}^*_{24} & \bar{q}^*_{34} \\ \bar{q}^*_{15} & \bar{q}^*_{25} & \bar{q}^*_{35} \\ \bar{q}^*_{16} & \bar{q}^*_{26} & \bar{q}^*_{36} \end{bmatrix} \begin{bmatrix} H_x + H^T_x \\ H_y + H^T_y \\ H_z + H^T_z \end{bmatrix} \quad (18)$$

An inherent assumption of lamination theory is that the laminate is in a state of plane stress. This assumption requires that σ_{zz}, σ_{yz}, and σ_{xz} equal zero throughout the laminate. Thus, eq. (18) reduces to,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} - \varepsilon_{xx}^{I} - \varepsilon_{xx}^{T} \\ \varepsilon_{yy} - \varepsilon_{yy}^{I} - \varepsilon_{yy}^{T} \\ 2\varepsilon_{xy} - 2\varepsilon_{xy}^{I} - 2\varepsilon_{xy}^{T} \end{bmatrix} - \begin{bmatrix} \hat{e}_{11} & \hat{e}_{21} & \hat{e}_{31} \\ \hat{e}_{12} & \hat{e}_{22} & \hat{e}_{32} \\ \hat{e}_{16} & \hat{e}_{26} & \hat{e}_{36} \end{bmatrix} \begin{bmatrix} E_x + E_x^T \\ E_y + E_y^T \\ E_z + E_z^T \end{bmatrix} - \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} & \hat{q}_{31} \\ \hat{q}_{12} & \hat{q}_{22} & \hat{q}_{32} \\ \hat{q}_{16} & \hat{q}_{26} & \hat{q}_{36} \end{bmatrix} \begin{bmatrix} H_x + H_x^T \\ H_y + H_y^T \\ H_z + H_z^T \end{bmatrix} \tag{19}$$

where $\bar{Q}_{ij}$ are components of the reduced stiffness matrix, $\hat{e}_{kj}$ are components of the reduced piezoelectric matrix, and $\hat{q}_{kj}$ are components of the reduced piezomagnetic matrix. It should be noted that, in general, $\bar{Q}_{ij}$ cannot be calculated from the simple expressions given in Jones (1975) and Herakovich (1998), which apply to orthotropic materials. Due to the presence of the 21 constants in the rotated effective stiffness matrix in eq. (18), these expressions become more complex. The procedure for determining the reduced stiffness, piezoelectric, and piezomagnetic terms in eq. (19) involves setting σ_{zz}, σ_{yz}, and σ_{xz} equal to zero in eq. (18) and simultaneously solving the third, fourth, and fifth equations of eq. (18) for the expressions, $\left(\varepsilon_{zz} - \varepsilon_{zz}^{I} - \varepsilon_{zz}^{T}\right)$, $\left(2\varepsilon_{yz} - 2\varepsilon_{yz}^{I} - 2\varepsilon_{yz}^{T}\right)$, and $\left(2\varepsilon_{xz} - 2\varepsilon_{xz}^{I} - 2\varepsilon_{xz}^{T}\right)$ in terms of the remaining field variables. Then, these expressions are substituted into the first, second, and sixth equations of eq. (18) and terms are grouped to identify the reduced matrix components of eq. (19). This procedure is clearly algebraically cumbersome and this author suggests performing the reduction numerically as part of a computer code implementation of the theory.

The next step follows classical lamination theory closely as the Kirchhoff-Love hypothesis (Jones, 1975; Herakovich, 1998) allows the total strains throughout the laminate to be related to the midplane strains, ε_{ij}^{0}, the midplane curvatures, κ_{ij}^{0}, and the through-thickness coordinate, z,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^{0} \\ \varepsilon_{yy}^{0} \\ 2\varepsilon_{xy}^{0} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx}^{0} \\ \kappa_{yy}^{0} \\ \kappa_{xy}^{0} \end{bmatrix} \tag{20}$$

The force and moment resultants for the laminate are given by,

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} dz \qquad \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} z\, dz \tag{21}$$

where t is the total thickness of the laminate. Substituting eq. (20) into eq. (19) and the result into eq. (21) yields,

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left\{ \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^{0} \\ \varepsilon_{yy}^{0} \\ 2\varepsilon_{xy}^{0} \end{bmatrix} + z\left[\bar{Q}_k\right] \begin{bmatrix} \kappa_{xx}^{0} \\ \kappa_{yy}^{0} \\ \kappa_{xy}^{0} \end{bmatrix} - \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^{I}(z) \\ \varepsilon_{yy}^{I}(z) \\ 2\varepsilon_{xy}^{I}(z) \end{bmatrix} - \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^{T} \\ \varepsilon_{yy}^{T} \\ 2\varepsilon_{xy}^{T} \end{bmatrix}_k - \left[\hat{e}_k\right] \begin{bmatrix} E_x(z) \\ E_y(z) \\ E_z(z) \end{bmatrix} - \left[\hat{e}_k\right] \begin{bmatrix} E_x^T \\ E_y^T \\ E_z^T \end{bmatrix}_k - \left[\hat{q}_k\right] \begin{bmatrix} H_x(z) \\ H_y(z) \\ H_z(z) \end{bmatrix} - \left[\hat{q}_k\right] \begin{bmatrix} H_x^T \\ H_y^T \\ H_z^T \end{bmatrix}_k \right\} dz \tag{22}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left\{ \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ 2\varepsilon_{xy}^0 \end{bmatrix} + z\left[\bar{Q}_k\right] \begin{bmatrix} \kappa_{xx}^0 \\ \kappa_{yy}^0 \\ \kappa_{xy}^0 \end{bmatrix} - \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^I(z) \\ \varepsilon_{yy}^I(z) \\ 2\varepsilon_{xy}^I(z) \end{bmatrix} - \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ 2\varepsilon_{xy}^T \end{bmatrix}_k - \left[\hat{e}_k\right] \begin{bmatrix} E_x(z) \\ E_y(z) \\ E_z(z) \end{bmatrix} - \left[\hat{e}_k\right] \begin{bmatrix} E_x^T \\ E_y^T \\ E_z^T \end{bmatrix}_k - \left[\hat{q}_k\right] \begin{bmatrix} H_x(z) \\ H_y(z) \\ H_z(z) \end{bmatrix} - \left[\hat{q}_k\right] \begin{bmatrix} H_x^T \\ H_y^T \\ H_z^T \end{bmatrix}_k \right\} z\, dz$$

where a subscript k indicates that a quantity is associated with layer number k and can thus vary from layer to layer. As indicated, the inelastic strain components can vary arbitrarily with the z coordinate, and thus must be integrated through the laminates thickness. Equation (22) also indicates that the electric and magnetic fields may vary through the laminates thickness. While this is true in general, the electric and magnetic fields must be admissible in that they must satisfy the LaPlace equation, and further must satisfy certain continuity conditions related to the electric potential and the magnetic potential. A technologically significant application of thermo-electro-magneto-elasto-plastic laminated plates involves applying different electric (or magnetic) potentials at the ply boundaries. With regard to the electric potentials, this would involve essentially attaching the two terminals of a battery to the top and bottom of a layer. This situation then establishes a constant electric field in the layer in question (as well as possibly the adjacent layers). Thus, the present theory will consider only electric and magnetic fields that are constant within each layer, but are permitted to vary between layers.

Employing the above assumption on the electric and magnetic fields and distributing the integrals in eq. (22) per layer results in,

$$\begin{aligned} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{N} \Bigg\{ & \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ 2\varepsilon_{xy}^0 \end{bmatrix} \int_{z_{k-1}}^{z_k} dz + \left[\bar{Q}_k\right] \begin{bmatrix} \kappa_{xx}^0 \\ \kappa_{yy}^0 \\ \kappa_{xy}^0 \end{bmatrix} \int_{z_{k-1}}^{z_k} z\, dz - \left[\bar{Q}_k\right] \int_{z_{k-1}}^{z_k} \begin{bmatrix} \varepsilon_{xx}^I(z) \\ \varepsilon_{yy}^I(z) \\ 2\varepsilon_{xy}^I(z) \end{bmatrix} dz - \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ 2\varepsilon_{xy}^T \end{bmatrix}_k \int_{z_{k-1}}^{z_k} dz \\ & - \left[\hat{e}_k\right] \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}_k \int_{z_{k-1}}^{z_k} dz - \left[\hat{e}_k\right] \begin{bmatrix} E_x^T \\ E_y^T \\ E_z^T \end{bmatrix}_k \int_{z_{k-1}}^{z_k} dz - \left[\hat{q}_k\right] \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}_k \int_{z_{k-1}}^{z_k} dz - \left[\hat{q}_k\right] \begin{bmatrix} H^T \\ H_y^T \\ H_z^T \end{bmatrix}_k \int_{z_{k-1}}^{z_k} dz \Bigg\} \end{aligned} \tag{23}$$

$$\begin{aligned} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{N} \Bigg\{ & \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ 2\varepsilon_{xy}^0 \end{bmatrix} \int_{z_{k-1}}^{z_k} z\, dz + \left[\bar{Q}_k\right] \begin{bmatrix} \kappa_{xx}^0 \\ \kappa_{yy}^0 \\ \kappa_{xy}^0 \end{bmatrix} \int_{z_{k-1}}^{z_k} z^2\, dz - \left[\bar{Q}_k\right] \int_{z_{k-1}}^{z_k} \begin{bmatrix} \varepsilon_{xx}^I(z) \\ \varepsilon_{yy}^I(z) \\ 2\varepsilon_{xy}^I(z) \end{bmatrix} z\, dz - \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ 2\varepsilon_{xy}^T \end{bmatrix}_k \int_{z_{k-1}}^{z_k} z\, dz \\ & - \left[\hat{e}_k\right] \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}_k \int_{z_{k-1}}^{z_k} z\, dz - \left[\hat{e}_k\right] \begin{bmatrix} E_x^T \\ E_y^T \\ E_z^T \end{bmatrix}_k \int_{z_{k-1}}^{z_k} z\, dz - \left[\hat{q}_k\right] \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}_k \int_{z_{k-1}}^{z_k} z\, dz - \left[\hat{q}_k\right] \begin{bmatrix} H_x^T \\ H_y^T \\ H_z^T \end{bmatrix}_k \int_{z_{k-1}}^{z_k} z\, dz \Bigg\} \end{aligned}$$

where N is the number of plies comprising the laminate and z_k refers to the z-coordinate of the top of ply number k (see Fig. 4). Integrating where appropriate, eq. (23) becomes the constitutive equation for the thermo-electro-magneto-elasto-plastic laminate,

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ 2\varepsilon_{xy}^0 \\ \kappa_{xx}^0 \\ \kappa_{yy}^0 \\ \kappa_{xy}^0 \end{bmatrix} - \begin{bmatrix} N_x^I \\ N_y^I \\ N_{xy}^I \\ M_x^I \\ M_y^I \\ M_{xy}^I \end{bmatrix} - \begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \\ M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} - \begin{bmatrix} N_x^E \\ N_y^E \\ N_{xy}^E \\ M_x^E \\ M_y^E \\ M_{xy}^E \end{bmatrix} - \begin{bmatrix} N_x^{ET} \\ N_y^{ET} \\ N_{xy}^{ET} \\ M_x^{ET} \\ M_y^{ET} \\ M_{xy}^{ET} \end{bmatrix} - \begin{bmatrix} N_x^M \\ N_y^M \\ N_{xy}^M \\ M_x^M \\ M_y^M \\ M_{xy}^M \end{bmatrix} - \begin{bmatrix} N_x^{MT} \\ N_y^{MT} \\ N_{xy}^{MT} \\ M_x^{MT} \\ M_y^{MT} \\ M_{xy}^{MT} \end{bmatrix} \tag{24}$$

where,

$$\mathbf{A} = \sum_{k=1}^{N} \left[\bar{Q}_k\right]\left(z_k - z_{k-1}\right) \qquad \mathbf{B} = \frac{1}{2}\sum_{k=1}^{N} \left[\bar{Q}_k\right]\left(z_k^2 - z_{k-1}^2\right) \qquad \mathbf{D} = \frac{1}{3}\sum_{k=1}^{N} \left[\bar{Q}_k\right]\left(z_k^3 - z_{k-1}^3\right) \tag{25}$$

$$\begin{bmatrix} N_x^I \\ N_y^I \\ N_{xy}^I \end{bmatrix} = \sum_{k=1}^{N} \left[\bar{Q}_k\right] \int_{z_{k-1}}^{z_k} \begin{bmatrix} \varepsilon_{xx}^I(z) \\ \varepsilon_{yy}^I(z) \\ 2\varepsilon_{xy}^I(z) \end{bmatrix} dz \qquad \begin{bmatrix} M_x^I \\ M_y^I \\ M_{xy}^I \end{bmatrix} = \sum_{k=1}^{N} \left[\bar{Q}_k\right] \int_{z_{k-1}}^{z_k} \begin{bmatrix} \varepsilon_{xx}^I(z) \\ \varepsilon_{yy}^I(z) \\ 2\varepsilon_{xy}^I(z) \end{bmatrix} z\, dz \tag{26}$$

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} = \sum_{k=1}^{N} \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ 2\varepsilon_{xy}^T \end{bmatrix}_k \left(z_k - z_{k-1}\right) \qquad \begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} = \frac{1}{2}\sum_{k=1}^{N} \left[\bar{Q}_k\right] \begin{bmatrix} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ 2\varepsilon_{xy}^T \end{bmatrix}_k \left(z_k^2 - z_{k-1}^2\right) \tag{27}$$

$$\begin{bmatrix} N_x^E \\ N_y^E \\ N_{xy}^E \end{bmatrix} = \sum_{k=1}^{N} \left[\hat{e}_k\right] \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}_k \left(z_k - z_{k-1}\right) \qquad \begin{bmatrix} M_x^E \\ M_y^E \\ M_{xy}^E \end{bmatrix} = \frac{1}{2}\sum_{k=1}^{N} \left[\hat{e}_k\right] \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}_k \left(z_k^2 - z_{k-1}^2\right) \tag{28}$$

$$\begin{bmatrix} N_x^{ET} \\ N_y^{ET} \\ N_{xy}^{ET} \end{bmatrix} = \sum_{k=1}^{N} \left[\hat{e}_k\right] \begin{bmatrix} E_x^T \\ E_y^T \\ E_z^T \end{bmatrix}_k \left(z_k - z_{k-1}\right) \qquad \begin{bmatrix} M_x^{ET} \\ M_y^{ET} \\ M_{xy}^{ET} \end{bmatrix} = \frac{1}{2}\sum_{k=1}^{N} \left[\hat{e}_k\right] \begin{bmatrix} E_x^T \\ E_y^T \\ E_z^T \end{bmatrix}_k \left(z_k^2 - z_{k-1}^2\right) \tag{29}$$

$$\begin{bmatrix} N_x^M \\ N_y^M \\ N_{xy}^M \end{bmatrix} = \sum_{k=1}^{N} \left[\hat{q}_k\right] \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}_k \left(z_k - z_{k-1}\right) \qquad \begin{bmatrix} M_x^M \\ M_y^M \\ M_{xy}^M \end{bmatrix} = \frac{1}{2}\sum_{k=1}^{N} \left[\hat{q}_k\right] \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}_k \left(z_k^2 - z_{k-1}^2\right) \tag{30}$$

$$\begin{bmatrix} N_x^{MT} \\ N_y^{MT} \\ N_{xy}^{MT} \end{bmatrix} = \sum_{k=1}^{N} \left[\hat{q}_k\right] \begin{bmatrix} H_x^T \\ H_y^T \\ H_z^T \end{bmatrix}_k \left(z_k - z_{k-1}\right) \qquad \begin{bmatrix} M_x^{MT} \\ M_y^{MT} \\ M_{xy}^{MT} \end{bmatrix} = \frac{1}{2}\sum_{k=1}^{N} \left[\hat{q}_k\right] \begin{bmatrix} H_x^T \\ H_y^T \\ H_z^T \end{bmatrix}_k \left(z_k^2 - z_{k-1}^2\right) \tag{31}$$

Thus, the simulated loading on the laminate can take the form of,

1. An admissible combination of force and moment resultants and midplane strains and curvatures.
2. A constant temperature change from a reference temperature.

3. A profile of electric and magnetic fields that is constant within each ply and satisfies continuity of normal electric and magnetic potential at the ply interfaces.

Armed with this applied loading, the unimposed force and moment resultants and midplane strains and curvatures are determined from eq. (24). From eq. (20) the in-plane strain field for the laminate (at the through-thickness integration points) is determined. Equation (19) provides the in-plane stresses, while eq. (18) gives the out-of-plane strain components. All of the field quantities are then transformed back to the local coordinates for each integration point, at which point eq. (16) is applicable. This is the thermo-electro-magneto-elasto-plastic constitutive equation for the homogenized material at a particular integration point. Thus, using the EMGMC theory presented in Section 3, the mechanical and electromagnetic field variables can be localized to the level of the individual constituents throughout the laminate (see Fig. 4). In the context of inelasticity, such localization is usually necessary as the inelastic constitutive models typically function on the level of homogeneous materials and require the local fields to determine the local inelastic strain increments. Once these are integrated to provide the local inelastic strains, EMGMC provides effective inelastic strains (as well as all other field variables) for the homogenized material (appearing in eq. (17)), which are transformed to the laminate coordinate system (as in eq. (18)). These homogenized and transformed inelastic strains are then integrated in eq. (26) to form the inelastic stress and moment resultants.

The above outlines a multi-scale approach to modeling thermo-electro-magneto-elasto-plastic composite structures. Herein, the particular structure that is considered is a laminated plate. Lamination theory provides the global or structural scale theory, while EMGMC provides the micro scale theory. In the presence of inelasticity, the localization and homogenization procedure discussed above must be repeated at each increment of the applied laminate-scale loading. An identical approach is possible for arbitrary thermo-electro-magneto-elasto-plastic composite structures by employing a finite element model as the structural theory. EMGMC would then provide the homogenized behavior of the composite at the integration points of the elements within the finite element mesh. This can be accomplished through user constitutive model routines that can function within some commercial finite element packages.

4. Results

In order to illustrate some of the unique features of the presented coupled micro/macro theory, the following results were generated using the theory as implemented within a computer code. It should be noted that the results are intended to display the theory's capabilities as opposed to addressing a practical structural design problem. Consider the symmetric $[0°/90°]_s$ laminate depicted in Fig. 5. The middle ply is a B/Al metal matrix composite (MMC) with a fixed fiber volume fraction of 0.25 oriented at 90°. Neither the boron fiber nor the aluminum matrix exhibits any electromagnetic-thermomechanical coupling, but the aluminum, as a metal, may be subject to inelastic deformation. The local inelastic constitutive response of the aluminum was thus modeled using the Bodner-Partom viscoplastic model (Chan et al., 1988), while the boron fiber was treated as linear elastic. The elastic properties for the boron fiber and the aluminum matrix are given in Table 1, while the Bodner-Partom viscoplastic model parameters for the aluminum matrix are given in Table 2. The middle ply B/Al composite was modeled using a four subcell repeating unit cell (i.e., $N_\alpha = 1$, $N_\beta = 2$, and $N_\gamma = 2$), wherein one subcell represents the boron fiber and the remaining three subcells represent the aluminum matrix. This is GMC's simplest geometric representation of a continuous fiber composite. The required through-thickness integration of the ply inelastic strains to determine the laminate inelastic force and moment resultants (see eq. (26)) was accomplished via second order Gauss quadrature.

The exterior plies of the laminate consist of continuous $BaTiO_3$ piezoelectric fibers in a $CoFe_2O_4$ piezomagnetic matrix forming a "smart" composite. These plies are treated as linear elastic (see Table 1 for the elastic properties). The poling direction for the $BaTiO_3$ piezoelectric fibers corresponds to the *x*-direction (i.e., along the fiber length), while the poling direction of the $CoFe_2O_4$ piezomagnetic matrix corresponds to the through-thickness *z*-direction. The ability to model a laminate such as this that contains materials with different poling directions highlights the utility of the generally anisotropic formulation employed. As in the MMC middle ply, a simple four subcell repeating unit cell was employed for the smart composite exterior plies.

Clearly, the above described laminate is symmetric. A second laminate configuration that results from simply reversing the poling direction of the $CoFe_2O_4$ piezomagnetic matrix in the bottom ply will also be considered. While this second configuration remains thermomechanically (and electrically) symmetric, due to the reverse in polarity, it is magnetically asymmetric. The electric and magnetic properties for the $BaTiO_3$ fibers and the $CoFe_2O_4$ matrix are given in Tables 3 and 4, respectively. Note that the "±" associated with the $CoFe_2O_4$ piezomagnetic coefficients refers to the positive and negative polarity cases described above. Also, neither $BaTiO_3$ nor $CoFe_2O_4$ exhibit coupling between their electric and magnetic responses as the magnetoelectric coefficients, a_{ij} in eq. (1), are zero for both materials.

The first interesting aspect of these laminates comes from the effective properties of the continuous $BaTiO_3/CoFe_2O_4$ plies. The effective piezoelectric, piezomagnetic, dielectric, magnetoelectric, and magnetic permeability matrices for this composite with a 0.25 fiber volume fraction and positive polarity (as determined by EMGMC) are,

$$\mathbf{e}^* = \begin{bmatrix} 4.22 & 0 & 0 \\ -1.59 & 0 & 0 \\ -1.55 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0432 \\ 0 & 0.0417 & 0 \end{bmatrix} \text{C/m}^2 \qquad \mathbf{q}^* = \begin{bmatrix} 0 & 0 & 263 \\ 0 & 0 & 255 \\ 0 & 0 & 325 \\ 0 & 271 & 0 \\ 426 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{N/Am}$$

$$\boldsymbol{\kappa}^* = \begin{bmatrix} 3.23 & 0 & 0 \\ 0 & 0.120 & 0 \\ 0 & 0 & 0.139 \end{bmatrix} \times 10^{-9} \text{ C/Vm} \qquad \mathbf{a}^* = \begin{bmatrix} 0 & 0 & 1.70 \\ 0 & 0 & 0 \\ -0.262 & 0 & 0 \end{bmatrix} \times 10^{-9} \text{ C/Am}$$

$$\boldsymbol{\mu}^* = \begin{bmatrix} -439 & 0 & 0 \\ 0 & -288 & 0 \\ 0 & 0 & 83.7 \end{bmatrix} \times 10^{-6} \text{ Ns}^2/\text{C}^2$$

Thus, despite the fact that neither constituent of the $BaTiO_3/CoFe_2O_4$ composite exhibits electro-magnetic coupling, through the matrix $\mathbf{a}^*$, the composite does. A similar observation was made by Li and Dunn (1998) and Aboudi (2000). If the polarity of the $CoFe_2O_4$ matrix is reversed such that it is aligned with the negative z-direction rather than the positive z-direction, the only manifestation in the effective composite properties is a sign reversal of the $\mathbf{q}^*$ and $\mathbf{a}^*$ matrices.

Considering the entire laminate described earlier (see Fig. 5), the response to applied loading in the form of a constant through-thickness magnetic field component, H_z, will be simulated. The remaining loading conditions require that the laminate is free of stress and moment resultants, free of electric fields, and free of the remaining magnetic field components. Further, the simulated laminate experiences no temperature change. Because the middle layer contains the time and history dependent aluminum matrix, the simulated loading must be applied incrementally. The chosen rate for this loading is 0.01 MA/m per second (i.e., 1×10^4 A/m per second). The global response of the symmetric laminate to the applied magnetic field is plotted in Fig. 6, while the response of the asymmetric laminate (with reversed magnetic polarity in the matrix of the bottom layer) is plotted in Fig. 7. For these simulations, the fiber volume fraction of each ply is 0.25.

Figure 6 indicates that, since the laminate is completely symmetric, the global response to the applied magnetic field involves only extension with no curvature. The laminate **ABD** matrix (see eq. 24) contains no bending-extension coupling terms (i.e., the matrix **B** contains only zeros), and the inelastic and magnetic moments integrate to zero through the laminate's thickness (see eqs. (26) and (30)). The inelastic and magnetic force resultants, on the other hand, are non-zero. The midplane strains, ε^0_{xx} and ε^0_{yy}, that arise from the applied magnetic field are plotted in Fig. 6. Also plotted is the average out-of-plane strain, ε_{zz}. This strain component does not participate directly in the laminate constitutive equation (24), but it can be determined via through-thickness averaging of the local out-of-plane strain components that must arise due to the plane stress requirement of lamination theory. Further, locally this strain component can be important as it may participate in the local inelastic constitutive model. The magnitude of the average out-of-plane strain is greater than that of the midplane strain components due to the large q^*_{33} component of the smart plies. The magnitude of ε^0_{yy} is the smallest of the plotted components because of the presence of the stiff continuous boron fibers (see Table 1) of the middle ply oriented in the y-direction. Finally, it is clear from the plots in Fig. 6 that at an applied magnetic field level of approximately 2 MA/m inelastic deformation of the middle B/Al layer begins to occur as the curves become nonlinear.

In addition to the midplane strain components shown in Fig. 6, an average electric displacement component, D_x, and an average magnetic flux density component, B_z, arise in the laminate. These quantities, like the out-of-plane strain can be calculated from the corresponding local quantities. The electric displacement component arises due to the non-zero a^*_{13} value in the smart plies, while the magnetic flux density component must be present due to the applied magnetic field (see eq. (1)).

In stark contrast to Fig. 6, Fig. 7 indicates that the magnetically asymmetric laminate experiences only bending with no midplane extension. This pure bending occurs despite the fact that the laminate is electro-mechanically symmetric; only the poling direction of the piezomagnetic matrix of the bottom ply has been reversed. As stated above, this reversal causes a sign reversal in the $\mathbf{q}^*$ and $\mathbf{a}^*$ matrices for the bottom ply, and a subsequent sign reversal in the reduced $\hat{\mathbf{q}}$ matrix for the ply (see eq. (19)). Thus, the signs of the $\hat{\mathbf{q}}$ matrices for the top and bottom plies of the laminate are opposite, and it is the magnetic force resultants that are zero while the magnetic moment resultants are non-zero (see eq. (30)). The average out-of-plane strain, ε_{zz}, as well as the average electric displacement component, D_x, is zero for this asymmetric laminate. The average magnetic flux density component, B_z, is identical to that which arose for the symmetric laminate (as $\boldsymbol{\mu}^*$ remains unchanged for each ply in the two cases).

In Fig. 7, the magnitude of κ_{yy} is smaller than that of κ_{xx} due to the presence of the stiff boron fibers of the middle ply oriented in the *y*-direction. The global manifestation of inelasticity in the middle MMC ply is less evident for the asymmetric laminate compared to its symmetric counterpart. Yielding begins at an applied magnetic field level of approximately 4 MA/m and causes only slight nonlinearity in the plotted curves of Fig. 7.

To further explore the inelastic deformation of the middle B/Al ply, the inelastic strain components for this layer are plotted in Figs. 8 and 9 for the two laminates. The plotted inelastic strain components are the global (unit cell) inelastic strains for the MMC in the middle ply. That is, they are the $\bar{\varepsilon}^I_{ij}$ components of eq. (16), representing the homogenized inelastic strains of the heterogeneous material. The asymmetric laminate inelastic strains represent those at the second-order Gauss integration point in the top half of the middle ply. The components at the other integration point in the middle ply, located in the lower half of the ply, are of the same magnitudes, but have opposite signs (due to the laminate's state of pure bending). Comparing Figs. 8 and 9 confirms that yielding of the B/Al ply occurs at a much lower applied magnetic field in the symmetric laminate as compared to the asymmetric laminate. Further, the inelastic strains rise more rapidly in the symmetric laminate. For both laminates, it is clear that the presence of the boron fibers suppresses inelastic strain in the direction of the fibers as the magnitude of ε_{11}^{I} is very small in both cases.

The effective inelastic strain, ε_{eff}^{I}, which is also plotted in Figs. 8 and 9, is a scalar quantity that can be employed to quantify the onset of yielding. This quantity is calculated via time integration of the effective inelastic strain increment, $d\varepsilon_{eff}^{I}$, where,

$$d\varepsilon_{eff}^{I} = \sqrt{2/3\,\varepsilon_{ij}^{I}\varepsilon_{ij}^{I}} \tag{32}$$

Employing a yield criterion for the middle ply of $\varepsilon_{eff}^{I} = 0.01\%$, a final study was performed for the two laminates. The fiber volume fraction of both exterior $BaTiO_3/CoFe_2O_4$ plies was varied, and the applied through-thickness magnetic field required to cause yielding of the B/Al ply was determined. The results for the symmetric and asymmetric laminates are plotted in Fig. 10. For all exterior ply fiber volume fractions, the symmetric laminate exhibits yielding at a lower applied magnetic field. As one would expect, the lowest magnetic field at yield for both laminates occurs for an exterior ply fiber volume fraction of zero. In this case, the laminate experiences the most severe deformation because the exterior plies contain only the piezomagnetic $CoFe_2O_4$ material. As the fiber volume fraction of the exterior plies rises, and they contain a greater percentage of the piezoelectric $BaTiO_3$ material, the deformation is less severe, and yielding is delayed. At an exterior ply fiber volume fraction of 1.0, yielding cannot occur because the laminate contains no piezomagnetic materials and thus does not respond mechanically to the applied magnetic field.

5. Conclusion

The framework for and equations of a coupled micro/macro theory for analysis of thermo-electro-magneto-elasto-plastic composite laminates has been presented. The theory employs the electro-magnetic generalized method of cells on the micro scale and classical lamination theory on the scale of the laminate. Both models have been kept generally anisotropic in terms of material thermo-mechanical behavior, electro-magnetic behavior, and coupled behavior. This provides the theory with a high level of flexibility in terms of potential analysis applications (e.g., a laminate of composite plies with constituents having different electro-magnetic poling directions). The presented theory includes inelastic effects, and by nature of its multi-scale characteristics, enables use of arbitrary viscoplastic constitutive models for the laminate constituents. Hence, the framework is suitable for the analysis of laminates composed of electro-magnetic, as well as metallic, phases. This point, as well as the utility of the generally anisotropic formulation, has been illustrated in the sample results that address yielding in a hybrid MMC/smart composite laminate.

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Appendix

The twelve global equations of the form of eq. (7) are,

$$\sum_\alpha d_\alpha\, ZI_{1,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)} + \sum_\alpha d_\alpha\, ZI_{1,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)} + \sum_\alpha d_\alpha\, ZI_{1,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)} + \sum_\alpha d_\alpha\, ZI_{1,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)} + \sum_\alpha d_\alpha\, ZI_{1,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)} + \sum_\alpha d_\alpha\, ZI_{1,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\alpha d_\alpha\, ZI_{1,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)} + \sum_\alpha d_\alpha\, ZI_{1,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)} + \sum_\alpha d_\alpha\, ZI_{1,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)} + \sum_\alpha d_\alpha\, ZI_{1,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)} + \sum_\alpha d_\alpha\, ZI_{1,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)} + \sum_\alpha d_\alpha\, ZI_{1,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$= d\bar{\varepsilon}_{11} - \sum_\alpha d_\alpha \varepsilon_{11}^{I(\alpha\beta\gamma)} - \sum_\alpha d_\alpha \varepsilon_{11}^{T(\alpha\beta\gamma)} \qquad \text{for all } \beta,\gamma$$

$$\sum_\beta h_\beta\, ZI_{2,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)} + \sum_\beta h_\beta\, ZI_{2,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)} + \sum_\beta h_\beta\, ZI_{2,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)} + \sum_\beta h_\beta\, ZI_{2,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)} + \sum_\beta h_\beta\, ZI_{2,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)} + \sum_\beta h_\beta\, ZI_{2,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\beta h_\beta\, ZI_{2,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)} + \sum_\beta h_\beta\, ZI_{2,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)} + \sum_\beta h_\beta\, ZI_{2,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)} + \sum_\beta h_\beta\, ZI_{2,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)} + \sum_\beta h_\beta\, ZI_{2,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)} + \sum_\beta h_\beta\, ZI_{2,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$= h\bar{\varepsilon}_{22} - \sum_\beta h_\beta \varepsilon_{22}^{I(\alpha\beta\gamma)} - \sum_\beta h_\beta \varepsilon_{22}^{T(\alpha\beta\gamma)} \qquad \text{for all } \alpha,\gamma$$

$$\sum_\gamma l_\gamma\, ZI_{3,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)} + \sum_\gamma l_\gamma\, ZI_{3,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)} + \sum_\gamma l_\gamma\, ZI_{3,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)} + \sum_\gamma l_\gamma\, ZI_{3,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)} + \sum_\gamma l_\gamma\, ZI_{3,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)} + \sum_\gamma l_\gamma\, ZI_{3,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\gamma l_\gamma\, ZI_{3,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)} + \sum_\gamma l_\gamma\, ZI_{3,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)} + \sum_\gamma l_\gamma\, ZI_{3,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)} + \sum_\gamma l_\gamma\, ZI_{3,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)} + \sum_\gamma l_\gamma\, ZI_{3,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)} + \sum_\gamma l_\gamma\, ZI_{3,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$= l\bar{\varepsilon}_{33} - \sum_\gamma l_\gamma \varepsilon_{33}^{I(\alpha\beta\gamma)} - \sum_\gamma l_\gamma \varepsilon_{33}^{T(\alpha\beta\gamma)} \qquad \text{for all } \alpha,\beta$$

$$\sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)} + \sum_\beta\sum_\gamma h_\beta l_\gamma\, ZI_{4,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$= 2hl\bar{\varepsilon}_{23} - 2\sum_\beta\sum_\gamma h_\beta l_\gamma\, \varepsilon_{23}^{I(\alpha\beta\gamma)} - 2\sum_\beta\sum_\gamma h_\beta l_\gamma\, \varepsilon_{23}^{T(\alpha\beta\gamma)} \qquad \text{for all } \alpha$$

$$\sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)} + \sum_\alpha\sum_\gamma d_\alpha l_\gamma\, ZI_{5,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$= 2dl\bar{\varepsilon}_{13} - 2\sum_\alpha\sum_\gamma d_\alpha l_\gamma\, \varepsilon_{13}^{I(\alpha\beta\gamma)} - 2\sum_\alpha\sum_\gamma d_\alpha l_\gamma\, \varepsilon_{13}^{T(\alpha\beta\gamma)} \qquad \text{for all } \beta$$

$$\sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)} + \sum_\alpha\sum_\beta d_\alpha h_\beta\, ZI_{6,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$= 2dh\bar{\varepsilon}_{12} - 2\sum_\alpha\sum_\beta d_\alpha h_\beta\, \varepsilon_{12}^{I(\alpha\beta\gamma)} - 2\sum_\alpha\sum_\beta d_\alpha h_\beta\, \varepsilon_{12}^{T(\alpha\beta\gamma)} \qquad \text{for all } \gamma$$

$$\sum_\alpha d_\alpha ZI_{7,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)}+\sum_\alpha d_\alpha ZI_{7,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)}+\sum_\alpha d_\alpha ZI_{7,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)}+\sum_\alpha d_\alpha ZI_{7,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)}+\sum_\alpha d_\alpha ZI_{7,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)}+\sum_\alpha d_\alpha ZI_{7,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\alpha d_\alpha ZI_{7,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)}+\sum_\alpha d_\alpha ZI_{7,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)}+\sum_\alpha d_\alpha ZI_{7,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)}+\sum_\alpha d_\alpha ZI_{7,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)}+\sum_\alpha d_\alpha ZI_{7,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)}+\sum_\alpha d_\alpha ZI_{7,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$=-d\bar{E}_1-\sum_\alpha d_\alpha E_1^{T(\alpha\beta\gamma)} \qquad \text{for all } \beta,\gamma$$

$$\sum_\beta h_\beta ZI_{8,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)}+\sum_\beta h_\beta ZI_{8,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)}+\sum_\beta h_\beta ZI_{8,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)}+\sum_\beta h_\beta ZI_{8,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)}+\sum_\beta h_\beta ZI_{8,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)}+\sum_\beta h_\beta ZI_{8,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\beta h_\beta ZI_{8,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)}+\sum_\beta h_\beta ZI_{8,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)}+\sum_\beta h_\beta ZI_{8,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)}+\sum_\beta h_\beta ZI_{8,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)}+\sum_\beta h_\beta ZI_{8,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)}+\sum_\beta h_\beta ZI_{8,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$=-h\bar{E}_2-\sum_\beta h_\beta E_2^{T(\alpha\beta\gamma)} \qquad \text{for all } \alpha,\gamma$$

$$\sum_\gamma l_\gamma ZI_{9,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)}+\sum_\gamma l_\gamma ZI_{9,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)}+\sum_\gamma l_\gamma ZI_{9,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)}+\sum_\gamma l_\gamma ZI_{9,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)}+\sum_\gamma l_\gamma ZI_{9,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)}+\sum_\gamma l_\gamma ZI_{9,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\gamma l_\gamma ZI_{9,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)}+\sum_\gamma l_\gamma ZI_{9,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)}+\sum_\gamma l_\gamma ZI_{9,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)}+\sum_\gamma l_\gamma ZI_{9,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)}+\sum_\gamma l_\gamma ZI_{9,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)}+\sum_\gamma l_\gamma ZI_{9,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$=-l\bar{E}_3-\sum_\gamma l_\gamma E_3^{T(\alpha\beta\gamma)} \qquad \text{for all } \alpha,\beta$$

$$\sum_\alpha d_\alpha ZI_{10,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)}+\sum_\alpha d_\alpha ZI_{10,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)}+\sum_\alpha d_\alpha ZI_{10,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)}+\sum_\alpha d_\alpha ZI_{10,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)}+\sum_\alpha d_\alpha ZI_{10,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)}+\sum_\alpha d_\alpha ZI_{10,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\alpha d_\alpha ZI_{10,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)}+\sum_\alpha d_\alpha ZI_{10,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)}+\sum_\alpha d_\alpha ZI_{10,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)}+\sum_\alpha d_\alpha ZI_{10,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)}+\sum_\alpha d_\alpha ZI_{10,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)}+\sum_\alpha d_\alpha ZI_{10,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$=-d\bar{H}_1-\sum_\alpha d_\alpha H_1^{T(\alpha\beta\gamma)} \qquad \text{for all } \beta,\gamma$$

$$\sum_\beta h_\beta ZI_{11,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)}+\sum_\beta h_\beta ZI_{11,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)}+\sum_\beta h_\beta ZI_{11,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)}+\sum_\beta h_\beta ZI_{11,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)}+\sum_\beta h_\beta ZI_{11,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)}+\sum_\beta h_\beta ZI_{11,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\beta h_\beta ZI_{11,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)}+\sum_\beta h_\beta ZI_{11,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)}+\sum_\beta h_\beta ZI_{11,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)}+\sum_\beta h_\beta ZI_{11,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)}+\sum_\beta h_\beta ZI_{11,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)}+\sum_\beta h_\beta ZI_{11,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$=-h\bar{H}_2-\sum_\beta h_\beta H_2^{T(\alpha\beta\gamma)} \qquad \text{for all } \alpha,\gamma$$

$$\sum_\gamma l_\gamma ZI_{12,1}^{(\alpha\beta\gamma)}\sigma_{11}^{(\beta\gamma)}+\sum_\gamma l_\gamma ZI_{12,2}^{(\alpha\beta\gamma)}\sigma_{22}^{(\alpha\gamma)}+\sum_\gamma l_\gamma ZI_{12,3}^{(\alpha\beta\gamma)}\sigma_{33}^{(\alpha\beta)}+\sum_\gamma l_\gamma ZI_{12,4}^{(\alpha\beta\gamma)}\sigma_{23}^{(\alpha)}+\sum_\gamma l_\gamma ZI_{12,5}^{(\alpha\beta\gamma)}\sigma_{13}^{(\beta)}+\sum_\gamma l_\gamma ZI_{12,6}^{(\alpha\beta\gamma)}\sigma_{12}^{(\gamma)}$$
$$+\sum_\gamma l_\gamma ZI_{12,7}^{(\alpha\beta\gamma)}D_1^{(\beta\gamma)}+\sum_\gamma l_\gamma ZI_{12,8}^{(\alpha\beta\gamma)}D_2^{(\alpha\gamma)}+\sum_\gamma l_\gamma ZI_{12,9}^{(\alpha\beta\gamma)}D_3^{(\alpha\beta)}+\sum_\gamma l_\gamma ZI_{12,10}^{(\alpha\beta\gamma)}B_1^{(\beta\gamma)}+\sum_\gamma l_\gamma ZI_{12,11}^{(\alpha\beta\gamma)}B_2^{(\alpha\gamma)}+\sum_\gamma l_\gamma ZI_{12,12}^{(\alpha\beta\gamma)}B_3^{(\alpha\beta)}$$
$$=-l\bar{H}_3-\sum_\gamma l_\gamma H_3^{T(\alpha\beta\gamma)} \qquad \text{for all } \alpha,\beta$$

The twelve equations of the form of eq. (14) are,

$$\bar{\sigma}_{11} = \frac{1}{hl}\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma \Bigg[d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{\varepsilon}_{33}$$
$$+2hl\sum_{i=1}^{N_\alpha} G\left(R_{\beta\gamma}, i + N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\beta\gamma}, i + N_2 + N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\beta\gamma}, i + N_2 + N_\alpha + N_\beta\right)\bar{\varepsilon}_{12}$$
$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i + N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{E}_3$$
$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_2 + N_\beta N_\gamma\right)\bar{H}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{H}_3$$
$$-\sum_{i=1}^{N_1} G\left(R_{\beta\gamma}, i\right) f^I(i) - \sum_{i=1}^{N_4} G\left(R_{\beta\gamma}, i\right) f^T(i)\Bigg]$$

$$\bar{\sigma}_{22} = \frac{1}{dl}\sum_{\alpha}\sum_{\gamma} d_\alpha l_\gamma \Bigg[d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}, i + N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}, i + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{\varepsilon}_{33}$$
$$+2hl\sum_{i=1}^{N_\alpha} G\left(R_{\alpha\gamma}, i + N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\alpha\gamma}, i + N_2 + N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\alpha\gamma}, i + N_2 + N_\alpha + N_\beta\right)\bar{\varepsilon}_{12}$$
$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}, i + N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}, i + N_1 + N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{E}_3$$
$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}, i + N_1 + N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}, i + N_1 + N_2 + N_\beta N_\gamma\right)\bar{H}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{H}_3$$
$$-\sum_{i=1}^{N_1} G\left(R_{\alpha\gamma}, i\right) f^I(i) - \sum_{i=1}^{N_4} G\left(R_{\alpha\gamma}, i\right) f^T(i)\Bigg]$$

$$\bar{\sigma}_{33} = \frac{1}{dh}\sum_{\alpha}\sum_{\beta} d_\alpha h_\beta \Bigg[d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}, i + N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}, i + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{\varepsilon}_{33}$$
$$+2hl\sum_{i=1}^{N_\alpha} G\left(R_{\alpha\beta}, i + N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\alpha\beta}, i + N_2 + N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\alpha\beta}, i + N_2 + N_\alpha + N_\beta\right)\bar{\varepsilon}_{12}$$
$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}, i + N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}, i + N_1 + N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{E}_3$$
$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}, i + N_1 + N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}, i + N_1 + N_2 + N_\beta N_\gamma\right)\bar{H}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{H}_3$$
$$-\sum_{i=1}^{N_1} G\left(R_{\alpha\beta}, i\right) f^I(i) - \sum_{i=1}^{N_4} G\left(R_{\alpha\beta}, i\right) f^T(i)\Bigg]$$

$$\bar{\sigma}_{23} = \frac{1}{d}\sum_{\alpha} d_{\alpha}\Bigg[d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\alpha, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\alpha, i + N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\alpha, i + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{\varepsilon}_{33}$$

$$+2hl\sum_{i=1}^{N_\alpha} G\left(R_\alpha, i + N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_\alpha, i + N_2 + N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_\alpha, i + N_2 + N_\alpha + N_\beta\right)\bar{\varepsilon}_{12}$$

$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\alpha, i + N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\alpha, i + N_1 + N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\alpha, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{E}_3$$

$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\alpha, i + N_1 + N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\alpha, i + N_1 + N_2 + N_\beta N_\gamma\right)\bar{H}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\alpha, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{H}_3$$

$$-\sum_{i=1}^{N_1} G\left(R_\alpha, i\right) f^I\left(i\right) - \sum_{i=1}^{N_4} G\left(R_\alpha, i\right) f^T\left(i\right)\Bigg]$$

$$\bar{\sigma}_{13} = \frac{1}{h}\sum_{\beta} h_{\beta}\Bigg[d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\beta, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\beta, i + N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\beta, i + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{\varepsilon}_{33}$$

$$+2hl\sum_{i=1}^{N_\alpha} G\left(R_\beta, i + N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_\beta, i + N_2 + N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_\beta, i + N_2 + N_\alpha + N_\beta\right)\bar{\varepsilon}_{12}$$

$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\beta, i + N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\beta, i + N_1 + N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\beta, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{E}_3$$

$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\beta, i + N_1 + N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\beta, i + N_1 + N_2 + N_\beta N_\gamma\right)\bar{H}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\beta, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{H}_3$$

$$-\sum_{i=1}^{N_1} G\left(R_\beta, i\right) f^I\left(i\right) - \sum_{i=1}^{N_4} G\left(R_\beta, i\right) f^T\left(i\right)\Bigg]$$

$$\bar{\sigma}_{12} = \frac{1}{l}\sum_{\gamma} l_{\gamma}\Bigg[d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\gamma, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\gamma, i + N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\gamma, i + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{\varepsilon}_{33}$$

$$+2hl\sum_{i=1}^{N_\alpha} G\left(R_\gamma, i + N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_\gamma, i + N_2 + N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_\gamma, i + N_2 + N_\alpha + N_\beta\right)\bar{\varepsilon}_{12}$$

$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\gamma, i + N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\gamma, i + N_1 + N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\gamma, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{E}_3$$

$$-d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_\gamma, i + N_1 + N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_\gamma, i + N_1 + N_2 + N_\beta N_\gamma\right)\bar{H}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_\gamma, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)\bar{H}_3$$

$$-\sum_{i=1}^{N_1} G\left(R_\gamma, i\right) f^I\left(i\right) - \sum_{i=1}^{N_4} G\left(R_\gamma, i\right) f^T\left(i\right)\Bigg]$$

$$
\begin{aligned}
\bar{D}_1 = \frac{1}{hl}\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma \Bigg[& d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}+N_1, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}+N_1, i+N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}+N_1, i+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{\varepsilon}_{33} \\
& +2hl\sum_{i=1}^{N_\alpha} G\left(R_{\beta\gamma}+N_1, i+N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\beta\gamma}+N_1, i+N_2+N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\beta\gamma}+N_1, i+N_2+N_\alpha+N_\beta\right)\bar{\varepsilon}_{12} \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}+N_1, i+N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}+N_1, i+N_1+N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}+N_1, i+N_1+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{E}_3 \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}+N_1, i+N_1+N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}+N_1, i+N_1+N_2+N_\beta N_\gamma\right)\bar{H}_2 \\
& -l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}+N_1, i+N_1+N_2+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{H}_3 - \sum_{i=1}^{N_4} G\left(R_{\beta\gamma}+N_1, i\right) f^T(i) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\bar{D}_2 = \frac{1}{dl}\sum_{\alpha}\sum_{\gamma} d_\alpha l_\gamma \Bigg[& d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}+N_1, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}+N_1, i+N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}+N_1, i+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{\varepsilon}_{33} \\
& +2hl\sum_{i=1}^{N_\alpha} G\left(R_{\alpha\gamma}+N_1, i+N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\alpha\gamma}+N_1, i+N_2+N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\alpha\gamma}+N_1, i+N_2+N_\alpha+N_\beta\right)\bar{\varepsilon}_{12} \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}+N_1, i+N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}+N_1, i+N_1+N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}+N_1, i+N_1+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{E}_3 \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}+N_1, i+N_1+N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}+N_1, i+N_1+N_2+N_\beta N_\gamma\right)\bar{H}_2 \\
& -l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}+N_1, i+N_1+N_2+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{H}_3 - \sum_{i=1}^{N_4} G\left(R_{\alpha\gamma}+N_1, i\right) f^T(i) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\bar{D}_3 = \frac{1}{dh}\sum_{\alpha}\sum_{\beta} d_\alpha h_\beta \Bigg[& d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}+N_1, i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}+N_1, i+N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}+N_1, i+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{\varepsilon}_{33} \\
& +2hl\sum_{i=1}^{N_\alpha} G\left(R_{\alpha\beta}+N_1, i+N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\alpha\beta}+N_1, i+N_2+N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\alpha\beta}+N_1, i+N_2+N_\alpha+N_\beta\right)\bar{\varepsilon}_{12} \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}+N_1, i+N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}+N_1, i+N_1+N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}+N_1, i+N_1+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{E}_3 \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}+N_1, i+N_1+N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}+N_1, i+N_1+N_2+N_\beta N_\gamma\right)\bar{H}_2 \\
& -l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}+N_1, i+N_1+N_2+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{H}_3 - \sum_{i=1}^{N_4} G\left(R_{\alpha\beta}+N_1, i\right) f^T(i) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\bar{B}_1 = \frac{1}{hl}\sum_{\beta}\sum_{\gamma} h_\beta l_\gamma \Bigg[& d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}+N_1+N_2,i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}+N_1+N_2,i+N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}+N_1+N_2,i+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{\varepsilon}_{33} \\
& +2hl\sum_{i=1}^{N_\alpha} G\left(R_{\beta\gamma}+N_1+N_2,i+N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\beta\gamma}+N_1+N_2,i+N_2+N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\beta\gamma}+N_1+N_2,i+N_2+N_\alpha+N_\beta\right)\bar{\varepsilon}_{12} \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}+N_1+N_2,i+N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}+N_1+N_2,i+N_1+N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}+N_1+N_2,i+N_1+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{E}_3 \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}+N_1+N_2,i+N_1+N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}+N_1+N_2,i+N_1+N_2+N_\beta N_\gamma\right)\bar{H}_2 \\
& -l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}+N_1+N_2,i+N_1+N_2+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{H}_3 - \sum_{i=1}^{N_4} G\left(R_{\beta\gamma}+N_1+N_2,i\right)f^T(i) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\bar{B}_2 = \frac{1}{dl}\sum_{\alpha}\sum_{\gamma} d_\alpha l_\gamma \Bigg[& d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}+N_1+N_2,i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{\varepsilon}_{33} \\
& +2hl\sum_{i=1}^{N_\alpha} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_2+N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_2+N_\alpha+N_\beta\right)\bar{\varepsilon}_{12} \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_1+N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_1+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{E}_3 \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_1+N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_1+N_2+N_\beta N_\gamma\right)\bar{H}_2 \\
& -l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}+N_1+N_2,i+N_1+N_2+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{H}_3 - \sum_{i=1}^{N_4} G\left(R_{\alpha\gamma}+N_1+N_2,i\right)f^T(i) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\bar{B}_3 = \frac{1}{dh}\sum_{\alpha}\sum_{\beta} d_\alpha h_\beta \Bigg[& d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}+N_1+N_2,i\right)\bar{\varepsilon}_{11} + h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}+N_1+N_2,i+N_\beta N_\gamma\right)\bar{\varepsilon}_{22} + l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}+N_1+N_2,i+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{\varepsilon}_{33} \\
& +2hl\sum_{i=1}^{N_\alpha} G\left(R_{\alpha\beta}+N_1+N_2,i+N_2\right)\bar{\varepsilon}_{23} + 2dl\sum_{i=1}^{N_\beta} G\left(R_{\alpha\beta}+N_1+N_2,i+N_2+N_\alpha\right)\bar{\varepsilon}_{13} + 2dh\sum_{i=1}^{N_\gamma} G\left(R_{\alpha\beta}+N_1+N_2,i+N_2+N_\alpha+N_\beta\right)\bar{\varepsilon}_{12} \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}+N_1+N_2,i+N_1\right)\bar{E}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}+N_1+N_2,i+N_1+N_\beta N_\gamma\right)\bar{E}_2 - l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}+N_1+N_2,i+N_1+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{E}_3 \\
& -d\sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\beta}+N_1+N_2,i+N_1+N_2\right)\bar{H}_1 - h\sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\beta}+N_1+N_2,i+N_1+N_2+N_\beta N_\gamma\right)\bar{H}_2 \\
& -l\sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\beta}+N_1+N_2,i+N_1+N_2+N_\beta N_\gamma+N_\alpha N_\gamma\right)\bar{H}_3 - \sum_{i=1}^{N_4} G\left(R_{\alpha\beta}+N_1+N_2,i\right)f^T(i) \Bigg]
\end{aligned}
$$

where,

$$R_{\beta\gamma} = \beta + N_\beta\left(\gamma - 1\right) \qquad R_{\alpha\gamma} = \alpha + N_\alpha\left(\gamma - 1\right) + N_\beta N_\gamma \qquad R_{\alpha\beta} = \alpha + N_\alpha\left(\beta - 1\right) + N_\beta N_\gamma + N_\alpha N_\gamma$$

$$R_\alpha = \alpha + N_2 \qquad R_\beta = \beta + N_2 + N_\alpha \qquad R_\gamma = \gamma + N_2 + N_\alpha + N_\beta$$

$$N_2 = N_\beta N_\gamma + N_\alpha N_\gamma + N_\alpha N_\beta \qquad N_1 = N_2 + N_\alpha + N_\beta + N_\gamma \qquad N_4 = N_1 + 2N_2$$

The components of the global electro-magneto-elastic coefficient matrix, $\mathbf{Z}^*$, in eq. (16) are,

$$C_{11}^* = \frac{d}{hl}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i\right) \qquad C_{12}^* = \frac{1}{l}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_\beta N_\gamma\right)$$

$$C_{13}^* = \frac{1}{h}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_\beta N_\gamma + N_\alpha N_\gamma\right) \qquad C_{14}^* = \sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\alpha} G\left(R_{\beta\gamma}, i + N_2\right)$$

$$C_{15}^* = \frac{d}{h}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\beta} G\left(R_{\beta\gamma}, i + N_2 + N_\alpha\right) \qquad C_{16}^* = \frac{d}{l}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\gamma} G\left(R_{\beta\gamma}, i + N_2 + N_\alpha + N_\beta\right)$$

$$e_{11}^* = \frac{d}{hl}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i + N_1\right) \qquad e_{21}^* = \frac{1}{l}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_\beta N_\gamma\right)$$

$$e_{31}^* = \frac{1}{h}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right) \qquad q_{11}^* = \frac{d}{hl}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_2\right)$$

$$q_{21}^* = \frac{1}{l}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\beta\gamma}, i + N_1 + N_2 + N_\beta N_\gamma\right) \qquad q_{31}^* = \frac{1}{h}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\beta\gamma}, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)$$

$$C_{22}^* = \frac{h}{dl}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}, i + N_\beta N_\gamma\right) \qquad C_{23}^* = \frac{1}{d}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}, i + N_\beta N_\gamma + N_\alpha N_\gamma\right)$$

$$C_{24}^* = \frac{h}{d}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\alpha} G\left(R_{\alpha\gamma}, i + N_2\right) \qquad C_{25}^* = \sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\beta} G\left(R_{\alpha\gamma}, i + N_2 + N_\alpha\right)$$

$$C_{26}^* = \frac{h}{l}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\gamma} G\left(R_{\alpha\gamma}, i + N_2 + N_\alpha + N_\beta\right) \qquad e_{12}^* = \frac{1}{l}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}, i + N_1\right)$$

$$e_{22}^* = \frac{h}{dl}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}, i + N_1 + N_\beta N_\gamma\right) \qquad e_{32}^* = \frac{1}{d}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}, i + N_1 + N_\beta N_\gamma + N_\alpha N_\gamma\right)$$

$$q_{12}^* = \frac{1}{l}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\beta N_\gamma} G\left(R_{\alpha\gamma}, i + N_1 + N_2\right) \qquad q_{22}^* = \frac{h}{dl}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\alpha N_\gamma} G\left(R_{\alpha\gamma}, i + N_1 + N_2 + N_\beta N_\gamma\right)$$

$$q_{32}^* = \frac{1}{d}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_\alpha N_\beta} G\left(R_{\alpha\gamma}, i + N_1 + N_2 + N_\beta N_\gamma + N_\alpha N_\gamma\right)$$

$$C_{33}^{*}=\frac{l}{dh}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\beta},i+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)\qquad C_{34}^{*}=\frac{l}{d}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}}G\left(R_{\alpha\beta},i+N_{2}\right)$$

$$C_{35}^{*}=\frac{l}{h}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\beta}}G\left(R_{\alpha\beta},i+N_{2}+N_{\alpha}\right)\qquad C_{36}^{*}=\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\gamma}}G\left(R_{\alpha\beta},i+N_{2}+N_{\alpha}+N_{\beta}\right)$$

$$e_{13}^{*}=\frac{1}{h}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\alpha\beta},i+N_{1}\right)\qquad e_{23}^{*}=\frac{1}{d}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\alpha\beta},i+N_{1}+N_{\beta}N_{\gamma}\right)$$

$$e_{33}^{*}=\frac{l}{dh}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\beta},i+N_{1}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)\qquad q_{13}^{*}=\frac{1}{h}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\alpha\beta},i+N_{1}+N_{2}\right)$$

$$q_{23}^{*}=\frac{1}{d}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\alpha\beta},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)\qquad q_{33}^{*}=\frac{l}{dh}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\beta},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$C_{44}^{*}=\frac{hl}{d}\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\alpha}}G\left(R_{\alpha},i+N_{2}\right)\qquad C_{45}^{*}=l\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\beta}}G\left(R_{\alpha},i+N_{2}+N_{\alpha}\right)$$

$$C_{46}^{*}=h\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\gamma}}G\left(R_{\alpha},i+N_{2}+N_{\alpha}+N_{\beta}\right)\qquad e_{14}^{*}=\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\alpha},i+N_{1}\right)$$

$$e_{24}^{*}=\frac{h}{d}\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\alpha},i+N_{1}+N_{\beta}N_{\gamma}\right)\qquad e_{34}^{*}=\frac{l}{d}\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha},i+N_{1}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$q_{14}^{*}=\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\alpha},i+N_{1}+N_{2}\right)\qquad q_{24}^{*}=\frac{h}{d}\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\alpha},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)$$

$$q_{34}^{*}=\frac{l}{d}\sum_{\alpha}d_{\alpha}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$C_{55}^{*}=\frac{dl}{h}\sum_{\beta}h_{\beta}\sum_{i=1}^{N_{\beta}}G\left(R_{\beta},i+N_{2}+N_{\alpha}\right)\qquad C_{56}^{*}=d\sum_{\beta}h_{\beta}\sum_{i=1}^{N_{\gamma}}G\left(R_{\beta},i+N_{2}+N_{\alpha}+N_{\beta}\right)$$

$$e_{15}^{*}=\frac{d}{h}\sum_{\beta}h_{\beta}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\beta},i+N_{1}\right)\qquad e_{25}^{*}=\sum_{\beta}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\beta},i+N_{1}+N_{\beta}N_{\gamma}\right)$$

$$e_{35}^{*}=\frac{l}{h}\sum_{\beta}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\beta},i+N_{1}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)\qquad q_{15}^{*}=\frac{d}{h}\sum_{\beta}h_{\beta}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\beta},i+N_{1}+N_{2}\right)$$

$$q_{25}^{*}=\sum_{\beta}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\beta},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)\qquad q_{35}^{*}=\frac{l}{h}\sum_{\beta}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\beta},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$C_{66}^{*}=\frac{dh}{l}\sum_{\gamma}l_{\gamma}\sum_{i=1}^{N_{\gamma}}G\left(R_{\gamma},i+N_{2}+N_{\alpha}+N_{\beta}\right)\qquad e_{16}^{*}=\frac{d}{l}\sum_{\gamma}l_{\gamma}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\gamma},i+N_{1}\right)$$

$$e_{26}^{*}=\frac{h}{l}\sum_{\gamma}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\gamma},i+N_{1}+N_{\beta}N_{\gamma}\right)\qquad e_{36}^{*}=\sum_{\gamma}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\gamma},i+N_{1}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$q_{16}^{*}=\frac{d}{l}\sum_{\gamma}l_{\gamma}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\gamma},i+N_{1}+N_{2}\right)\qquad q_{26}^{*}=\frac{h}{l}\sum_{\gamma}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\gamma},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)$$

$$q_{36}^{*}=\sum_{\gamma}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\gamma},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$-\kappa_{11}^{*}=\frac{d}{hl}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\beta\gamma}+N_{1},i+N_{1}\right)\qquad -\kappa_{12}^{*}=\frac{1}{l}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\beta\gamma}+N_{1},i+N_{1}+N_{\beta}N_{\gamma}\right)$$

$$-\kappa_{13}^{*}=\frac{1}{h}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\beta\gamma}+N_{1},i+N_{1}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)\qquad -a_{11}^{*}=\frac{d}{hl}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\beta\gamma}+N_{1},i+N_{1}+N_{2}\right)$$

$$-a_{12}^{*}=\frac{1}{l}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\beta\gamma}+N_{1},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)\qquad -a_{13}^{*}=\frac{1}{h}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\beta\gamma}+N_{1},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$-\kappa_{22}^{*}=\frac{h}{dl}\sum_{\alpha}\sum_{\gamma}d_{\alpha}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\alpha\gamma}+N_{1},i+N_{1}+N_{\beta}N_{\gamma}\right)\qquad -\kappa_{23}^{*}=\frac{1}{d}\sum_{\alpha}\sum_{\gamma}d_{\alpha}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\gamma}+N_{1},i+N_{1}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$-a_{21}^{*}=\frac{1}{l}\sum_{\alpha}\sum_{\gamma}d_{\alpha}l_{\gamma}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\alpha\gamma}+N_{1},i+N_{1}+N_{2}\right)\qquad -a_{22}^{*}=\frac{h}{dl}\sum_{\alpha}\sum_{\gamma}d_{\alpha}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\alpha\gamma}+N_{1},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)$$

$$-a_{23}^{*}=\frac{1}{d}\sum_{\alpha}\sum_{\gamma}d_{\alpha}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\gamma}+N_{1},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$-\kappa_{33}^{*}=\frac{l}{dh}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\beta}+N_{1},i+N_{1}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)\qquad -a_{31}^{*}=\frac{1}{h}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\alpha\beta}+N_{1},i+N_{1}+N_{2}\right)$$

$$-a_{32}^{*}=\frac{1}{d}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\alpha\beta}+N_{1},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)\qquad -a_{33}^{*}=\frac{l}{dh}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\beta}+N_{1},i+N_{1}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$-\mu_{11}^{*}=\frac{d}{hl}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\beta}N_{\gamma}}G\left(R_{\beta\gamma}+N_{1}+N_{2},i+N_{1}+N_{2}\right)\qquad -\mu_{12}^{*}=\frac{1}{l}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\beta\gamma}+N_{1}+N_{2},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)$$

$$-\mu_{13}^{*}=\frac{1}{h}\sum_{\beta}\sum_{\gamma}h_{\beta}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\beta\gamma}+N_{1}+N_{2},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$-\mu_{22}^{*}=\frac{h}{dl}\sum_{\alpha}\sum_{\gamma}d_{\alpha}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\gamma}}G\left(R_{\alpha\gamma}+N_{1}+N_{2},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}\right)\qquad -\mu_{23}^{*}=\frac{1}{d}\sum_{\alpha}\sum_{\gamma}d_{\alpha}l_{\gamma}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\gamma}+N_{1}+N_{2},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

$$-\mu_{33}^{*}=\frac{l}{dh}\sum_{\alpha}\sum_{\beta}d_{\alpha}h_{\beta}\sum_{i=1}^{N_{\alpha}N_{\beta}}G\left(R_{\alpha\beta}+N_{1}+N_{2},i+N_{1}+N_{2}+N_{\beta}N_{\gamma}+N_{\alpha}N_{\gamma}\right)$$

The components of the global inelastic strain vector, $\bar{\boldsymbol{\varepsilon}}^I$, global thermal strain vector, $\bar{\boldsymbol{\varepsilon}}^T$, global thermo-electric field vector, $\bar{\mathbf{E}}^T$, and the global thermo-magnetic field vector, $\bar{\mathbf{H}}^T$, in eq. (16) are,

$$\begin{bmatrix} \bar{\varepsilon}_{11}^I \\ \bar{\varepsilon}_{22}^I \\ \bar{\varepsilon}_{33}^I \\ 2\bar{\varepsilon}_{23}^I \\ 2\bar{\varepsilon}_{13}^I \\ 2\bar{\varepsilon}_{12}^I \end{bmatrix} = \left[\mathbf{C}^*\right]^{-1} \begin{bmatrix} \frac{1}{hl}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_1} G\left(R_{\beta\gamma},i\right) f^I(i) \\ \frac{1}{dl}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_1} G\left(R_{\alpha\gamma},i\right) f^I(i) \\ \frac{1}{dh}\sum_\alpha\sum_\beta d_\alpha h_\beta \sum_{i=1}^{N_1} G\left(R_{\alpha\beta},i\right) f^I(i) \\ \frac{1}{d}\sum_\alpha d_\alpha \sum_{i=1}^{N_1} G\left(R_\alpha,i\right) f^I(i) \\ \frac{1}{h}\sum_\beta h_\beta \sum_{i=1}^{N_1} G\left(R_\beta,i\right) f^I(i) \\ \frac{1}{l}\sum_\gamma l_\gamma \sum_{i=1}^{N_1} G\left(R_\gamma,i\right) f^I(i) \end{bmatrix}$$

$$\begin{bmatrix} \bar{\varepsilon}_{11}^T \\ \bar{\varepsilon}_{22}^T \\ \bar{\varepsilon}_{33}^T \\ 2\bar{\varepsilon}_{23}^T \\ 2\bar{\varepsilon}_{13}^T \\ 2\bar{\varepsilon}_{12}^T \\ -\bar{E}_1^T \\ -\bar{E}_2^T \\ -\bar{E}_3^T \\ -\bar{H}_1^T \\ -\bar{H}_2^T \\ -\bar{H}_3^T \end{bmatrix} = \left[\mathbf{Z}^*\right]^{-1} \begin{bmatrix} \frac{1}{hl}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_4} G\left(R_{\beta\gamma},i\right) f^T(i) \\ \frac{1}{dl}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_4} G\left(R_{\alpha\gamma},i\right) f^T(i) \\ \frac{1}{dh}\sum_\alpha\sum_\beta d_\alpha h_\beta \sum_{i=1}^{N_4} G\left(R_{\alpha\beta},i\right) f^T(i) \\ \frac{1}{d}\sum_\alpha d_\alpha \sum_{i=1}^{N_4} G\left(R_\alpha,i\right) f^T(i) \\ \frac{1}{h}\sum_\beta h_\beta \sum_{i=1}^{N_4} G\left(R_\beta,i\right) f^T(i) \\ \frac{1}{l}\sum_\gamma l_\gamma \sum_{i=1}^{N_4} G\left(R_\gamma,i\right) f^T(i) \\ \frac{1}{hl}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_4} G\left(R_{\beta\gamma}+N_1,i\right) f^T(i) \\ \frac{1}{dl}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_4} G\left(R_{\alpha\gamma}+N_1,i\right) f^T(i) \\ \frac{1}{dh}\sum_\alpha\sum_\beta d_\alpha h_\beta \sum_{i=1}^{N_4} G\left(R_{\alpha\beta}+N_1,i\right) f^T(i) \\ \frac{1}{hl}\sum_\beta\sum_\gamma h_\beta l_\gamma \sum_{i=1}^{N_4} G\left(R_{\beta\gamma}+N_1+N_2,i\right) f^T(i) \\ \frac{1}{dl}\sum_\alpha\sum_\gamma d_\alpha l_\gamma \sum_{i=1}^{N_4} G\left(R_{\alpha\gamma}+N_1+N_2,i\right) f^T(i) \\ \frac{1}{dh}\sum_\alpha\sum_\beta d_\alpha h_\beta \sum_{i=1}^{N_4} G\left(R_{\alpha\beta}+N_1+N_2,i\right) f^T(i) \end{bmatrix}$$

Table 1. Elastic properties for the boron fiber, aluminum matrix, $BaTiO_3$ piezoelectric fiber, and $CoFe_2O_4$ piezomagnetic matrix (Aboudi, 2000). Note: properties are given in principal material coordinates of plies (see Fig. 5) – $BaTiO_3$ properties are given for an x_1 poling direction, $CoFe_2O_4$ properties are given for an x_3 poling direction.

	E_{11} (GPa)	E_{22} (GPa)	E_{33} (GPa)	ν_{12}	ν_{23}	G_{23} (GPa)	G_{13} (GPa)	G_{12} (GPa)
Boron	400.00	400.00	400.00	0.20	0.20	166.67	166.67	166.67
Aluminum	71.93	71.93	71.93	0.33	0.33	27.04	27.04	27.04
$BaTiO_3$	111.93	116.33	116.33	0.321	0.307	44.50	43.00	43.00
$CoFe_2O_4$	154.57	154.57	143.57	0.368	0.398	45.30	45.30	56.49

Table 2. Bodner-Partom viscoplastic properties for the aluminum matrix (Aboudi, 1991).

	D_0 (s^{-1})	Z_0 (MPa)	Z_1 (MPa)	m	n
Aluminum	1000	65	150	50	10

Table 3. Non-zero electric properties of the $BaTiO_3$ piezoelectric fiber and $CoFe_2O_4$ piezomagnetic matrix (Aboudi, 2000). Note: properties are given in principal material coordinates of plies (see Fig. 5) – $BaTiO_3$ properties are given for an x_1 poling direction, $CoFe_2O_4$ properties are given for an x_3 poling direction.

	e_{11} (C/m^2)	e_{12} (C/m^2)	e_{13} (C/m^2)	e_{26} (C/m^2)	e_{35} (C/m^2)	κ_{11} (10^{-9}C/Vm)	κ_{22} (10^{-9}C/Vm)	κ_{33} (10^{-9}C/Vm)
$BaTiO_3$	18.6	-4.4	-4.4	11.6	11.6	12.6	12.6	11.2
$CoFe_2O_4$	0	0	0	0	0	0.08	0.08	0.093

Table 4. Non-zero magnetic properties of the $BaTiO_3$ piezoelectric fiber and $CoFe_2O_4$ piezomagnetic matrix (Aboudi, 2000). Note: properties are given in principal material coordinates of plies (see Fig. 5) – $BaTiO_3$ properties are given for an x_1 poling direction, $CoFe_2O_4$ properties are given for an x_3 poling direction. The "±" symbols refer to positive or negative x_3-direction magnetic polarity.

	q_{15} (N/Am)	q_{14} (N/Am)	q_{31} (N/Am)	q_{32} (N/Am)	q_{33} (N/Am)	μ_{11} ($10^{-6}Ns^2/C^2$)	μ_{22} ($10^{-6}Ns^2/C^2$)	μ_{33} ($10^{-6}Ns^2/C^2$)
$BaTiO_3$	0	0	0	0	0	10	5	5
$CoFe_2O_4$	±550	±550	±580.3	±580.3	±699.7	-590	-590	157

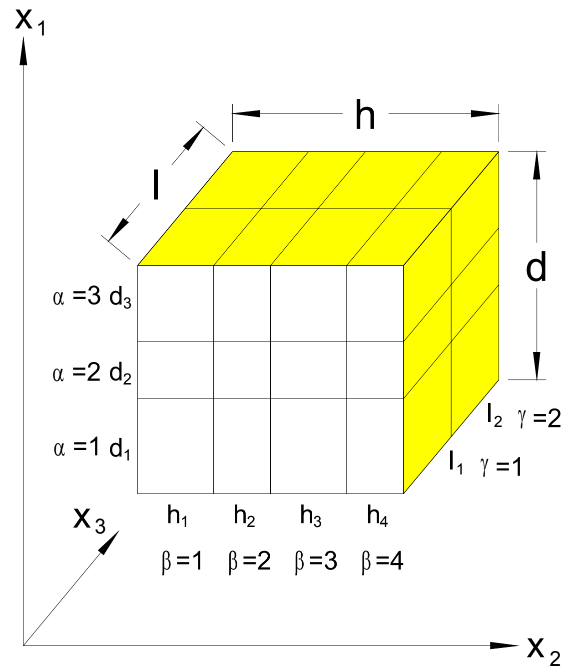

Fig. 1. Three-dimensional generalized method of cells geometry with $N_\alpha = 3,\ N_\beta = 4,$ and $N_\gamma = 2$.

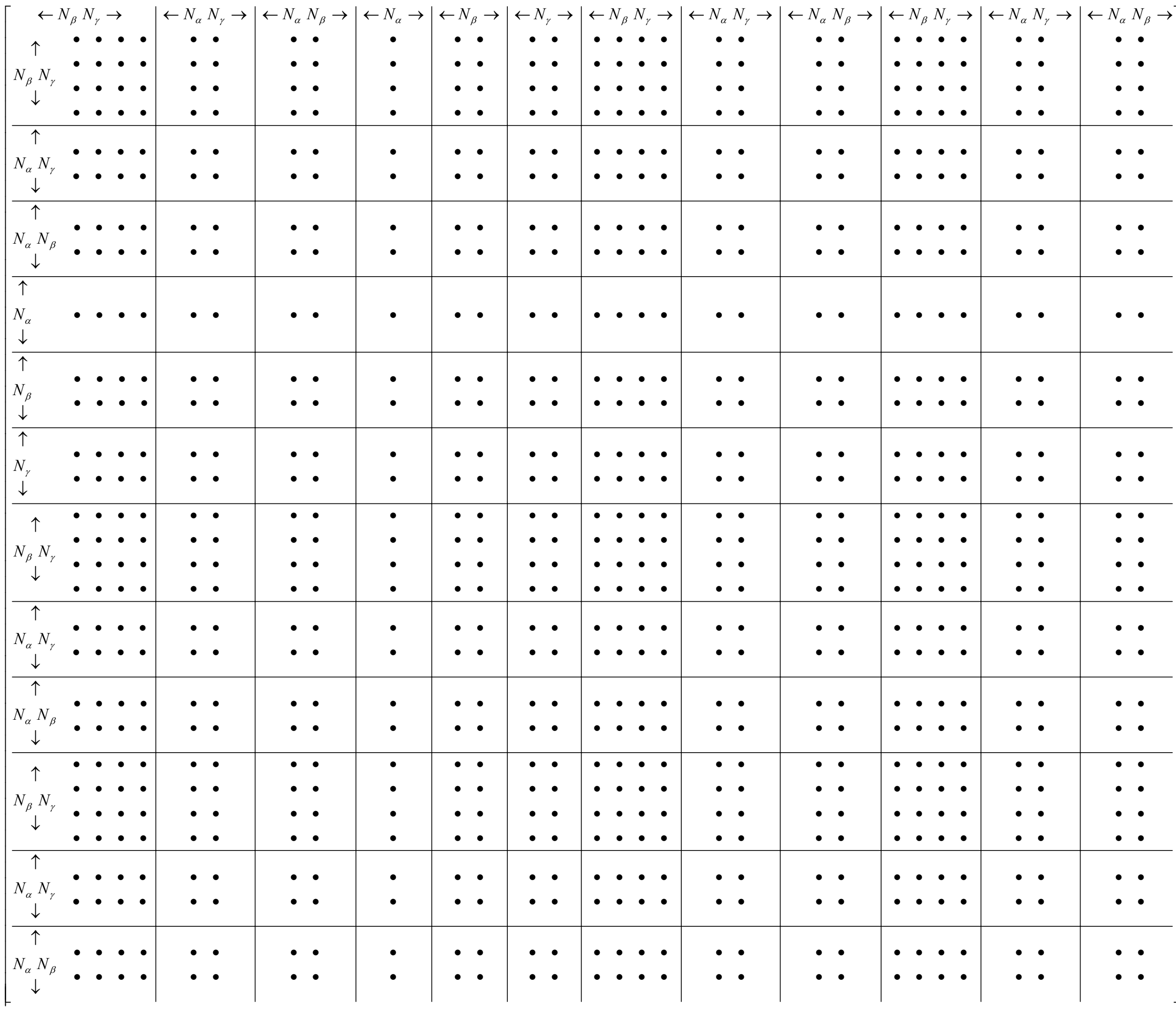

Fig. 2. Form of the $\tilde{\mathbf{G}}$ matrix in the reformulation of EMGMC.

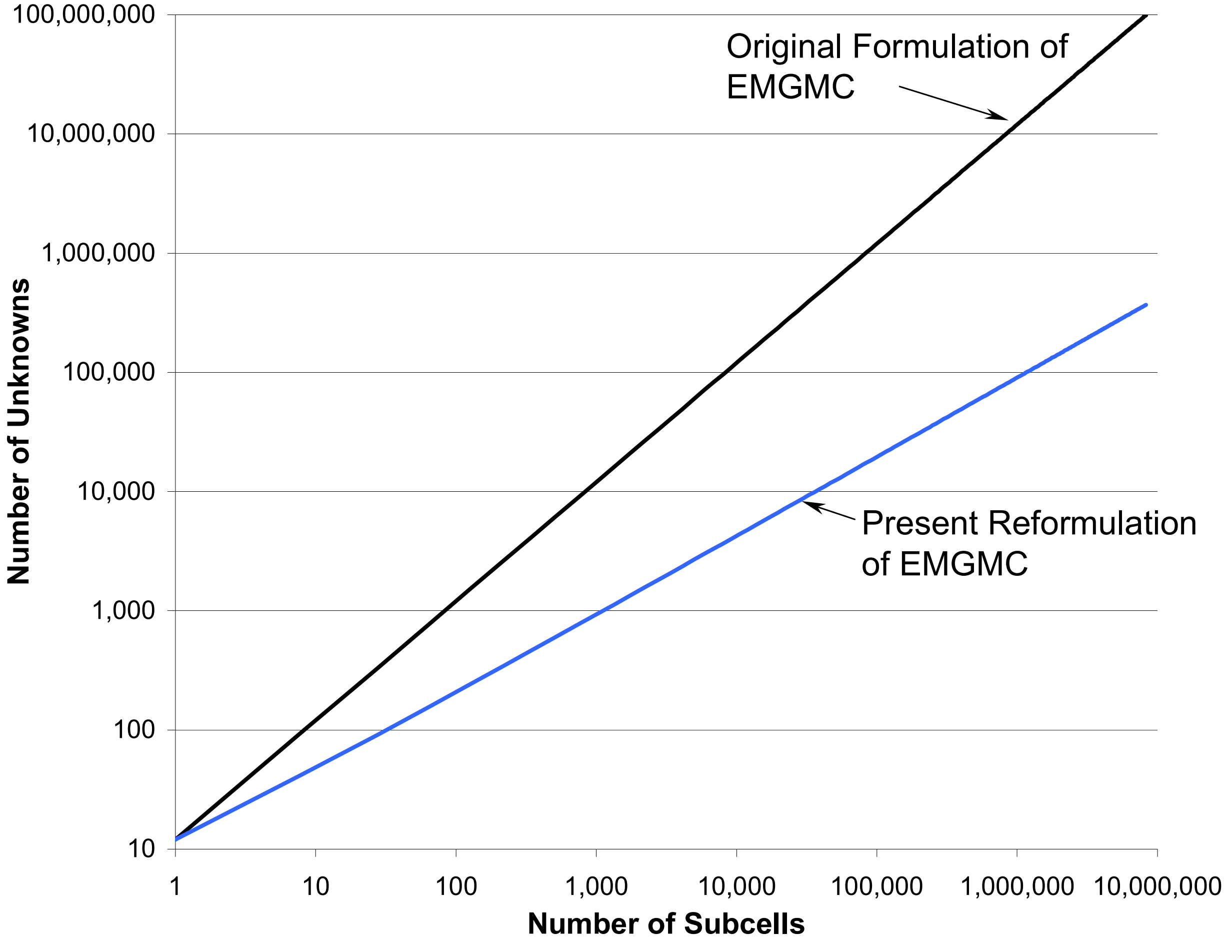

Fig. 3. Plot of the number of unknown quantities vs. number of subcells comprising the repeating unit cell in the original formulation and present reformulation of EMGMC for the case where $N_\alpha = N_\beta = N_\gamma$.

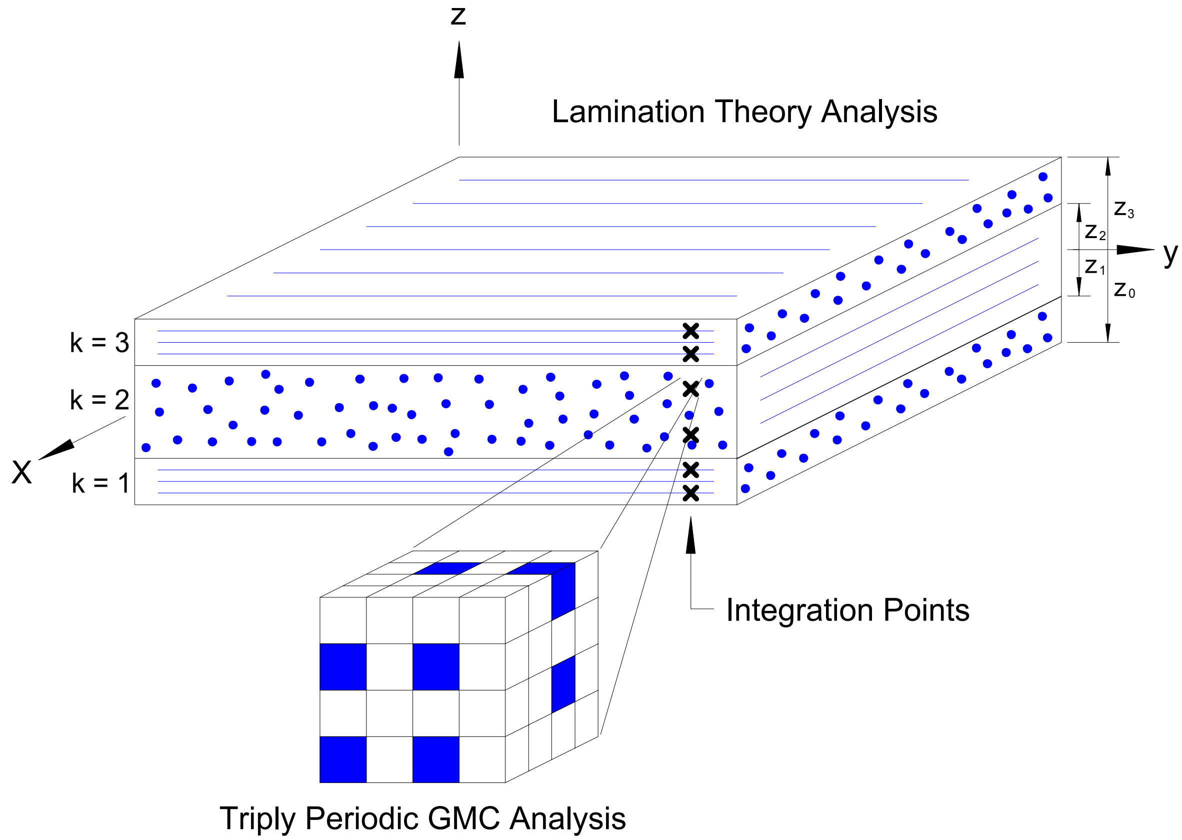

Fig. 4. Laminated plate geometry and schematic of the multi-scale framework.

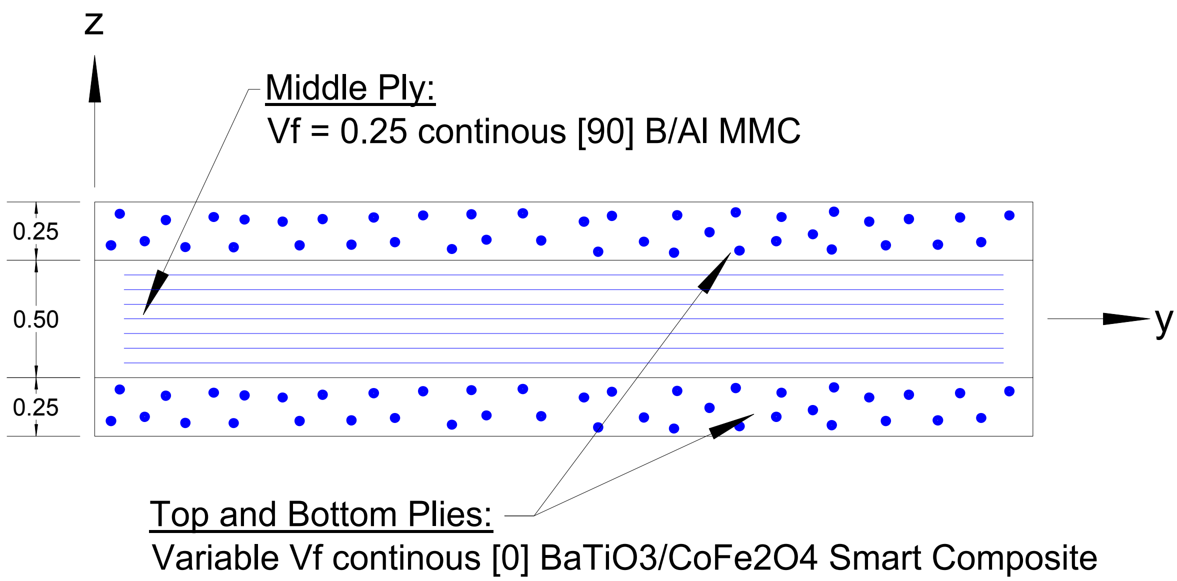

Fig. 5. A $[0°/90°]_s$ hybrid $BaTiO_3/CoFe_2O_4$ – B/Al smart/MMC laminate.

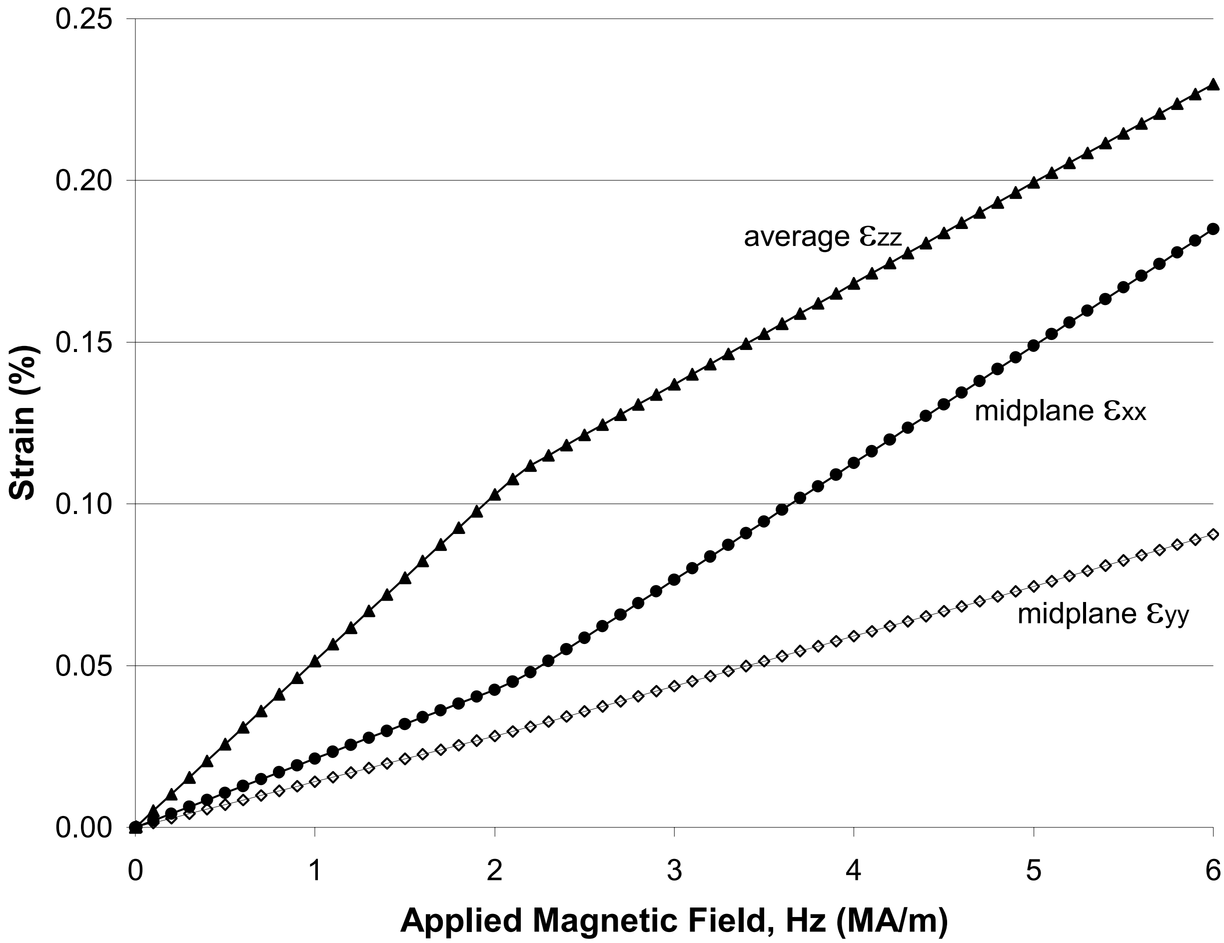

Fig. 6. Global strain response of a symmetric $[0°/90°]_s$ hybrid $BaTiO_3/CoFe_2O_4$ – B/Al smart/MMC laminate to an applied through-thickness magnetic field. The fiber volume fraction of each ply is 0.25.

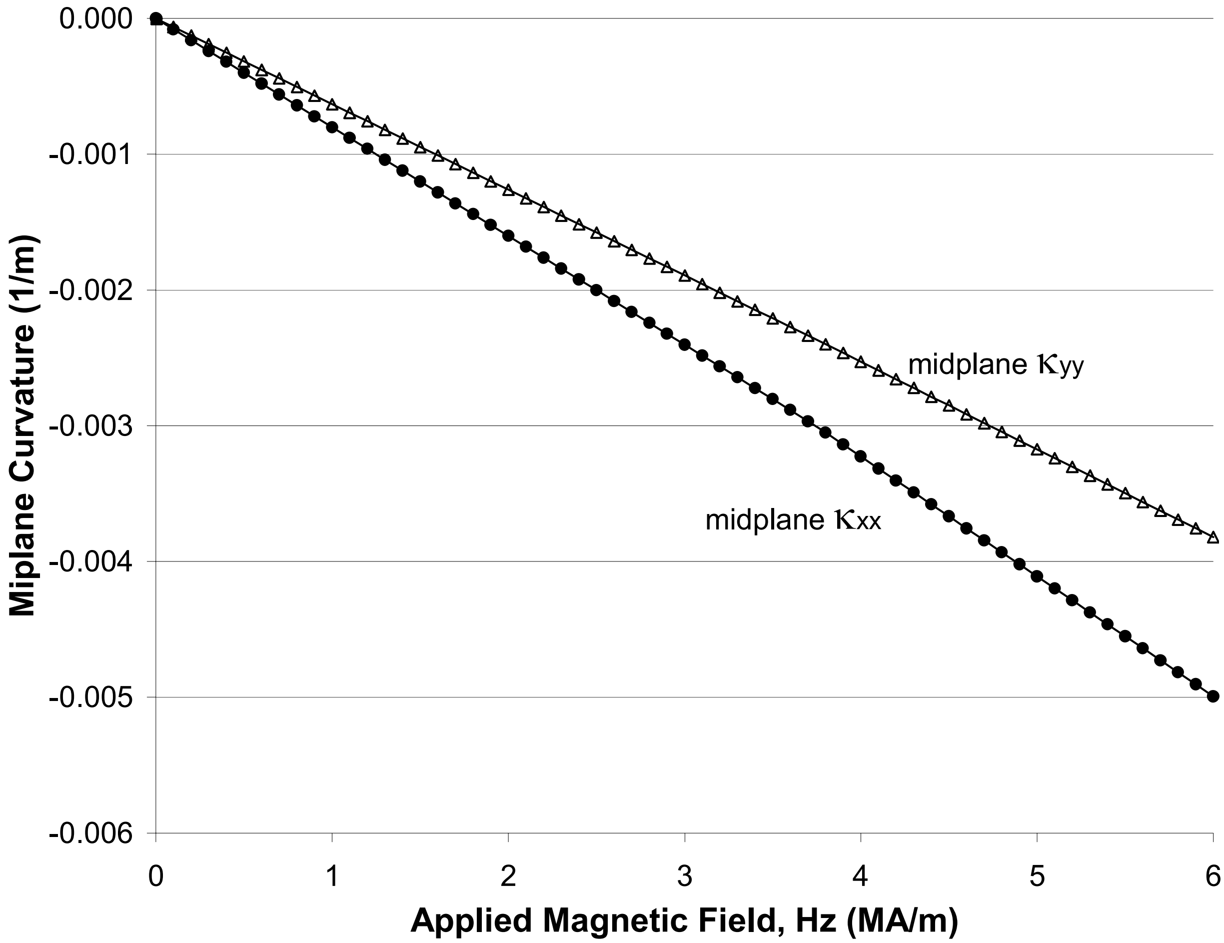

Fig. 7. Global curvature response of a magnetically asymmetric [0°/90°]$_s$ hybrid $BaTiO_3/CoFe_2O_4$ – B/Al smart/MMC laminate to an applied through-thickness magnetic field. The fiber volume fraction of each ply is 0.25.

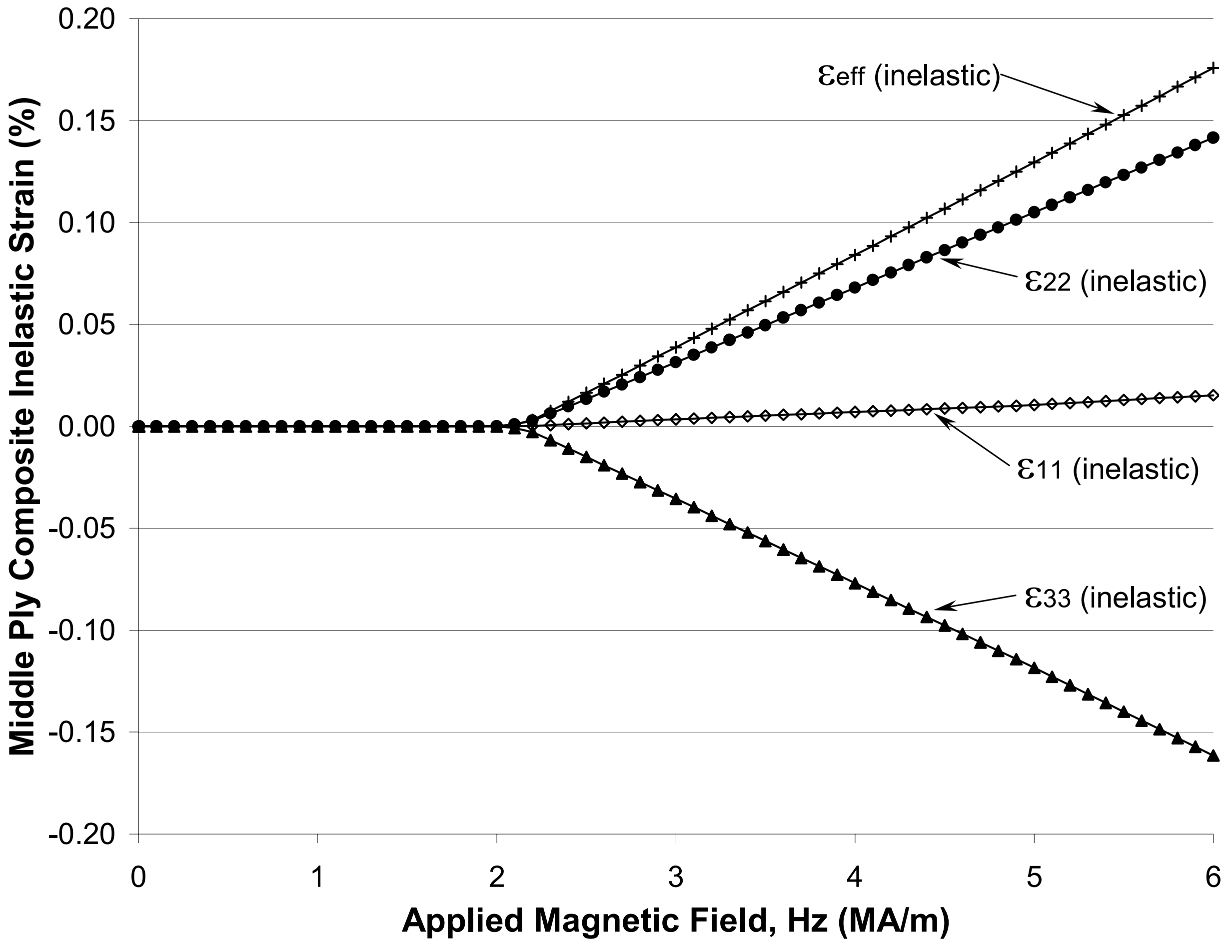

Fig. 8. Inelastic strain response of the middle ply of a symmetric $[0°/90°]_s$ hybrid $BaTiO_3/CoFe_2O_4$ – B/Al smart/MMC laminate to an applied through-thickness magnetic field. The fiber volume fraction of each ply is 0.25. Note that the inelastic strain components are given in the local coordinate system of the middle 90° B/Al ply. The effective inelastic strain, ε^I_{eff}, is determined through time-integration of the effective inelastic strain increment, i.e., $\varepsilon^I_{eff} = \int d\varepsilon^I_{eff} = \int\sqrt{2/3\, d\varepsilon^I_{ij}\, d\varepsilon^I_{ij}}$.

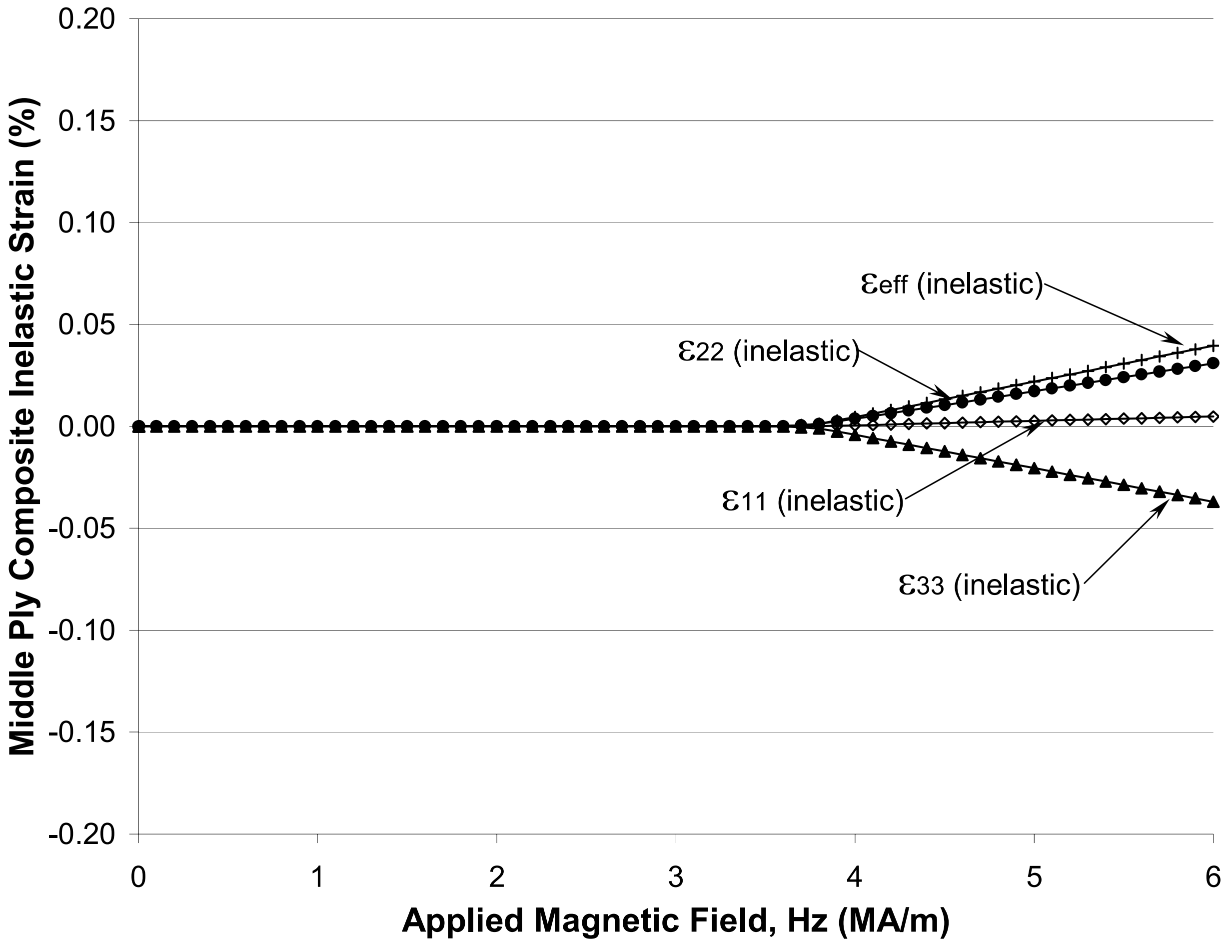

Fig. 9. Inelastic strain response of the middle ply of a magnetically asymmetric [0°/90°]$_s$ hybrid $BaTiO_3/CoFe_2O_4$ – B/Al smart/MMC laminate to an applied through-thickness magnetic field. The fiber volume fraction of each ply is 0.25. Note that the inelastic strain components are given in the local coordinate system of the middle 90° B/Al ply. The effective inelastic strain, ε^I_{eff}, is determined through time-integration of the effective inelastic strain increment, i.e., $\varepsilon^I_{eff} = \int d\varepsilon^I_{eff} = \int \sqrt{2/3\, d\varepsilon^I_{ij}\, d\varepsilon^I_{ij}}$.

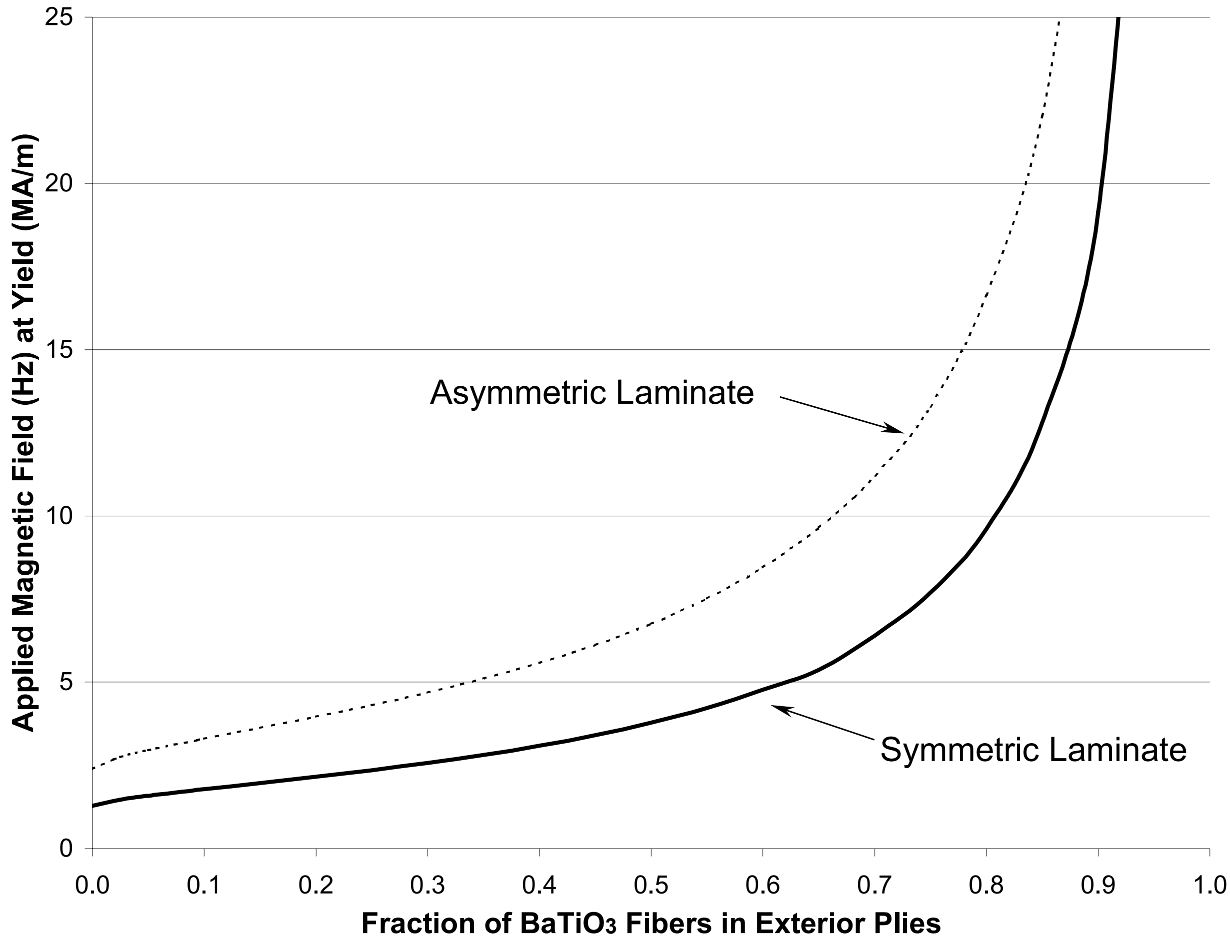

Fig. 10. Applied through-thickness magnetic field required to cause yielding in the middle ply of a $[0°/90°]_s$ hybrid $BaTiO_3/CoFe_2O_4$ – B/Al smart/MMC laminate as a function of the fiber volume fraction of the exterior smart plies.

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13. ABSTRACT ***(Maximum 200 words)***

This paper presents a micro/macro theory for determining the coupled thermo-electro-magneto-elasto-plastic behavior of arbitrary composite laminates. Two models are considered. The first is the electro-magnetic generalized method of cells micromechanics model (EMGMC) introduced by Aboudi in NASA/CR—2000-209787. Herein, EMGMC has been completely reformulated to improve its computational efficiency and has been extended to admit arbitrary anisotropic local material behavior (in terms of the thermal response, mechanical response, electric response, magnetic response, as well as the coupling behavior) and inelastic local material behavior. The second model is classical lamination theory, which has also been extended for arbitrary anisotropic material behavior and electro-magnetic effects. The end result is a coupled theory that employs EMGMC to provide the homogenized behavior of the composite plies that constitute the thermo-electro-magnetic laminate. Sample results that illustrate many of the unique aspects of the theory are presented.

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